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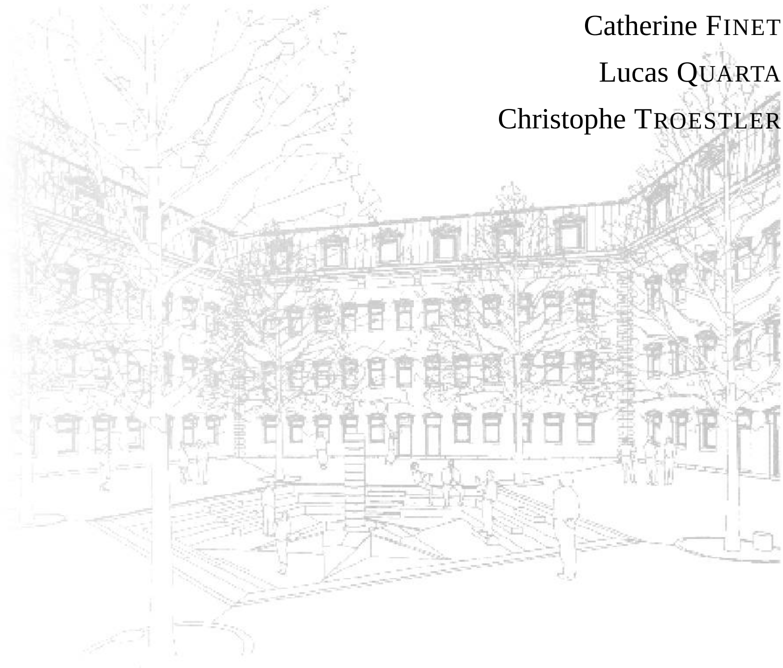

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Abstract. In the context of vector-valued extensions of variational principles we are dealing with functions taking values in a Banach space partially ordered by a closed convex pointed cone. We introduce and study a new notion of semi-continuity connected with the order and we improve the vector-valued extensions of Deville-Godefroy-Zizler perturbed minimization principle.

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Introduction

We are interested in vector-valued variational principles (or perturbed minimization principles). Let us describe the context. Let Z be a class of real-valued functions on a complete metric space X and $f : X \to \mathbb{R}$ be a bounded below lower semi-continuous function. Is it possible to perturb f by an element g of Z in such a way that the new function $f+g$ attains its minimum? These principles have been extensively studied (see for example the well-known theorems of Bishop-Phelps [4], Ekeland [12, 13] and also [6, 8, 10, 11]). More generally one can consider a function $f : X \to Y$ with Y a real Banach space partially ordered by a closed convex pointed cone (see for example [2, 9, 14, 15, 16, 18, 24, 25, 26]). In the vector-valued case there are several possible extensions of the "scalar" notion of lower semi-continuity, let us mention lower semi-continuity (lsc) and quasi lower semi-continuity (q-lsc) [26]. Under the hypothesis that the interior of the ordering cone is non-empty, R. Deville and C. Finet proved a vector-valued version of the Deville-Godefroy-Zizler perturbed minimization principle for bounded below q-lsc functions [9]. In [14], this result is obtained without the above hypothesis on the cone but only for lsc functions. All these proofs are using a scalarization process (see Remark 29). In this paper, we introduce a new notion of lower semi-continuity weaker than the two others. We call it order lower semi-continuity (o-lsc) because it links the norm topology with the partial order of Y. We then establish a vector-valued version of the Deville-Godefroy-Zizler perturbed minimization principle for o-lsc functions (Theorem 27) which improves the above two. The interest here is in the proof which does not use any scalarization process.

Section 1 of the paper is devoted to some preliminaries, particularly the definitions of lower semi-continuity and quasi lower semi-continuity and the link between these notions.

In section 2, we introduce the notion of order lower semi-continuity and we describe the relation to the other notions of lower semi-continuity. Lower semi-continuity implies order lower semi-continuity (Proposition 3). The converse is not true even when Y is finite dimensional. However, when f is an order-bounded function and each order interval of Y is compact, order lower semi-continuity implies lower semi-continuity (Proposition 14). For real-valued functions, the three notions of lower semi-continuity coincide. In general q-lsc does not imply o-lsc. We introduce a new property for a partially ordered Banach space called Monotone Bounds Property (MBP) and prove the implication for Banach spaces with MBP (Proposition 7). Many Banach spaces have MBP, for example when the interior of the ordering cone is non-empty (to our knowledge, the only case in which the optimization problem for q-lsc functions has been investigated), $\mathrm{L}^p(\mathbb{R}, \mu)$ where μ is a measure and $1 \leqslant p \leqslant \infty$, $\mathcal{C}_0(\mathbb{R}^n)$, every Banach lattice (and in particular every Banach space with unconditional basis, endowed with the natural order) but not any finite dimensional space has MBP. When f is an order-bounded function, we get the equivalence of the three notions for Y with MBP and such that each order interval is compact, in particular for finite dimensional Y (Corollaries 10 and 15). This section shows the complexity of the situation for the vector-valued case in comparison with the scalar one.

A summary of the different relationships is given (see page 19).

In section 3, we study the sum of (quasi-, order-) lower semi-continuous functions. The sum of two lsc functions is lsc [26] but this is not true for q-lsc or o-lsc functions. However, when the Banach space Y has MBP and is such that the order intervals are compact the sum of two o-lsc bounded below functions is o-lsc (Lemma 22). In any case, the sum of a continuous function and a o-lsc function is o-lsc (Lemma 21).

Section 4 is devoted to a vector-valued version of the Deville-Godefroy-Zizler perturbed minimization principle (Theorem 27). We introduce the notion of ε-infimal points of a non-empty subset of Y. We prove the existence and localization of such points for bounded below subsets of Y (Proposition 26) and use it to establish our extension without scalarization process. As corollaries, we get a vector-valued extension of Ekeland variational principle (Corollary 31) and of Borwein-Preiss smooth perturbed minimization principle (Corollary 32).

1 Preliminaries and notation

Throughout this paper, X and Y are two real Banach spaces and Y is partially ordered by a closed convex pointed cone K. No assumption is required on the interior of K.

For any elements $y, z \in Y$, we will write $y \leqslant z$ whenever $z - y \in K$. The set $[y, z] := \{x \in Y : y \leqslant x \leqslant z\}$ is called the *order interval* between y and z. We say that a sequence $(y_n) \subset Y$ is *non-increasing* and we use the notation $y_n \searrow$ whenever, for all n, $y_{n+1} \leqslant y_n$. The ball of center x_0 and radius r (resp. in X) will be denoted by $\mathrm{B}(x_0, r)$ (resp. $\mathrm{B}_X(x_0, r)$).

Let S be a non-empty subset of Y. We denote respectively by $\operatorname{int} S$ and $\overline{S}$, the interior and closure of S. Let us recall the associated algebraic notions [17]. The *algebraic interior*, $\operatorname{cor} S$, of S is the set of points y such that for each $z \in Y$ there exists some real number $\lambda_z > 0$ with $y + \lambda z \in S$ for all λ in $[0, \lambda_z]$. An element $y \in Y$ is called *linearly accessible* from S if there exists some $z \in S$, $z \neq y$, such that $\lambda z + (1 - \lambda)y \in S$ for all λ in $]0, 1]$. The union of S and the set of all linearly accessible elements from S is called the *algebraic* *closure* of S and its denoted by $\operatorname{lin} S$. If S is convex and $\operatorname{int} S \neq \emptyset$, then $\operatorname{int} S = \operatorname{cor} S$ and $\overline{S} = \operatorname{lin} S$. We denote by $\operatorname{Aff} S$ the *affine hull* of S.

Let f be a function from X to Y. In [5] and [26], the authors introduced the following two notions of lower semi-continuity:

- f is said to be *lower semi-continuous* (lsc) at $x_0 \in X$ iff, for each neighborhood V of $f(x_0)$ in Y, there exists a neighborhood U of x_0 in X such that $f(U) \subset V + K$.
- f is said to be *quasi lower semi-continous* (q-lsc) at $x_0 \in X$ iff, for each $b \in Y$ such that $b \not\geqslant f(x_0)$, there exists a neighborhood U of x_0 such that $b \not\geqslant f(x)$ for each x in U.

A function f is (resp. *quasi*-) *upper semi-continuous*, usc for short (resp. q-usc), if $-f$ is lsc (resp. q-lsc). A function f is lsc (resp. quasi-lsc) if f is lsc (resp. q-lsc) at each point of X.

Let us give some well-known facts concerning these notions (see [5] and [26]).

- A function f is lsc at x_0 iff $\lim_{x \to x_0} d\big(f(x), f(x_0) + K\big) = 0$.
- A function f is q-lsc iff for each b in Y, the set $\{f \leqslant b\} := \{x \in X : f(x) \leqslant b\}$ is closed in X.
- A lsc function at x_0 is q-lsc at x_0.

The notions of lsc and q-lsc coincide for real-valued functions, but it is not the case in general: if we take $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, then the function $f : \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(x) = \begin{cases} \big(-1, 1/|x|\big) & \text{if } x \neq 0 \\ (0,0) & \text{if } x = 0 \end{cases} \tag{1}$$

is q-lsc but not lsc at 0.

A function f is said to be *bounded below* (resp. *above*) if there exists some b in Y such that $b \leqslant f(x)$ (resp. $f(x) \leqslant b$) for all $x \in Y$.

A function f is said to be *order-bounded* if it is both bounded below and above.

2 Order lower semi-continuity

In this section, we introduce a new notion of lower semi-continuity called order lower semi-continuity because of its relationship with the order. We compare this notion with the previous ones and we give some examples. We also study the relationship between continuity and order lower and upper semi-continuity.

Definition 1. A function $f : X \to Y$ is said to be *order lower semi-continuous* (o-lsc) at $x_0 \in X$ iff, for each sequence $(x_n) \subset X$ converging to x_0 for which there exists a sequence $(\varepsilon_n) \subset Y$ converging to 0 such that the sequence $(f(x_n) + \varepsilon_n)$ is non-increasing, there exists a sequence $(g_n) \subset Y$ converging to 0 such that $f(x_0) \leqslant f(x_n) + g_n$ for all n.

A function $f : X \to Y$ is said to be *order upper semi-continuous* (o-usc) if $-f$ is o-lsc.

The definition of o-lsc can be expressed more briefly as follows:

$$x_n \to x_0 \text{ and } f(x_n) + \mathrm{o}(1) \searrow \implies f(x_0) \leqslant f(x_n) + \mathrm{o}(1).$$

It is easy to prove :

Lemma 2. *A function $f : X \to Y$ is lsc at x_0 iff $(x_n \to x_0) \Rightarrow \big(f(x_0) \leqslant f(x_n) + \mathrm{o}(1)\big)$.*

It follows :

Proposition 3. *If f is lsc at x_0 then f is o-lsc at x_0.*

In general, for vector-valued functions, o-lsc does not imply lsc. For example, let us take again the function f defined from $\mathbb{R}$ to $\mathbb{R}^2$ (endowed with the natural ordering cone $\mathbb{R}^2_+$) by (1). This function is o-lsc but not lsc at 0. We will prove that, for real-valued functions, the three notions of semi-continuity coincide.

Lemma 4. *Let f be a function from X to Y. Then f is lsc at x_0 (resp. o-lsc) iff from any sequence (x_n) converging to $x_0 \in X$ (resp. with $f(x_n) + \mathrm{o}(1) \searrow$) one can extract a subsequence (x'_n) such that $f(x_0) \leqslant f(x'_n) + \mathrm{o}(1)$.*

PROOF. We will only deal with the assertion for lsc functions, the one for o-lsc being similar. The 'only if' part is obvious. For the 'if' part, let us suppose by contradiction that f is not lsc. Therefore there exists some $\varepsilon > 0$ and a sequence $x_n \to x_0$ in X such that,

$$\text{(2)} \qquad \text{for all } n, \quad f(x_n) \notin \mathrm{B}(f(x_0), \varepsilon) + K.$$

By hypothesis, one can extract a subsequence $(x'_n) \subset (x_n)$ such that $f(x_0) \leqslant f(x'_n) + \mathrm{o}(1)$. Thus $f(x'_n) \geqslant f(x_0) + \mathrm{o}(1) \in \mathrm{B}(f(x_0), \varepsilon)$ when n is large. This contradicts (2). □

Corollary 5. *For real-valued functions the three notions of lower semi-continuity are equivalent.*

PROOF. Let f be a real-valued function defined on X. As recalled above, q-lsc and lsc are identical properties for real-valued functions. So, all we have to prove is that if f is o-lsc at $x_0 \in X$ then f is lsc at x_0. We will use the characterization of lsc functions given by Lemma 2. Let $(x_n) \subset X$ be a sequence converging to x_0. If $(f(x_n))$ is bounded, one can extract a subsequence $(x'_n) \subset (x_n)$ such that $(f(x'_n))$ converges in $\mathbb{R}$: $\exists \alpha \in \mathbb{R}$, $f(x'_n) + \mathrm{o}(1) = \alpha$. Since f is o-lsc at x_0, $f(x_0) \leqslant f(x'_n) + \mathrm{o}(1)$. On the other hand, if $(f(x_n))$ is unbounded, either one can extract a subsequence such that $f(x'_n) \searrow -\infty$ or $f(x'_n) \to +\infty$ and, in the two cases, the same conclusion holds: $f(x_0) \leqslant f(x'_n) + \mathrm{o}(1)$. By Lemma 4, f is lsc. □

Now, we want to prove the implication "q-lsc ⇒ o-lsc" for a quite large family of partially ordered Banach spaces.

Definition 6 (Monotone Bounds Property). One says that (Y, K) has the *monotone bounds property* (MBP) iff any sequence $(y_n) \subset Y$ converging to 0 has a subsequence $(y'_n) \subset (y_n)$ for which there exists a non-increasing sequence $(\overline{y}_n) \subset Y$ converging to 0 such that for all n, $y'_n \leqslant \overline{y}_n$.

Let us note that not any (Y, K), even finite dimensional, has MBP. Indeed, let $Y = \mathbb{R}^2$, $K = \{(x, 0) \in \mathbb{R}^2 : x \geqslant 0\}$ and take, for example, $y_n = (0, 1/n)$ for all n in $\mathbb{N}_0$. Let us give some examples of (Y, K) with MBP.

- **If K has a non-empty interior then (Y, K) has MBP.**

Indeed, let $e_0 \in \operatorname{int} K$ and $\delta > 0$ be such that $\mathrm{B}(e_0, \delta) \subset K$. Then $\mathrm{B}(0, 1) \subset -e_0/\delta + K$ and by symmetry we also have $\mathrm{B}(0, 1) \subset e_0/\delta - K$. So, by letting $e = e_0/\delta$, we have, for each $y \in Y$, $\pm y \leqslant \|y\| e$. If $(y_n) \subset Y$ is a sequence converging to 0 it suffices to take $\overline{y}_n := \left(\sup_{m \geqslant n} \|y_m\|\right) e$.

It follows that ℓ^∞ and $\mathrm{L}^\infty(\Omega, \mu)$, where (Ω, μ) is a measurable space, endowed with the natural ordering cone have MBP.

Let us remark that when $\operatorname{int} K \neq \varnothing$, MBP is equivalent to the following property: each sequence converging to 0 in Y has an order-bounded subsequence.

- **If (Y, K) is a Banach lattice then (Y, K) has MBP.**

Indeed, let (y_n) be a sequence in Y converging to 0. Since $(|y_n|)$ is also a sequence converging to 0, up to a subsequence, we can assume that $\sum_{n=1}^{\infty} \big\| |y_n| \big\|_Y$ is finite. Let us define $(\overline{y}_n)$ by $\overline{y}_n := \sum_{m \geqslant n} |y_m|$ for all n. It is clear that $(\overline{y}_n)$ is a non-increasing sequence converging to 0 in Y and $y_n \leqslant \overline{y}_n$ for all n.

It follows that every Banach space with unconditional basis endowed with the natural order associated to the basis has MBP. In particular, ℓ^p ($1 \leqslant p < +\infty$) and c_0, endowed with the natural orders, satisfy MBP. Also, $\mathrm{L}^p(\Omega, \mu)$ ($1 \leqslant p < +\infty$), where (Ω, μ) is a measurable space, endowed with the order associated to the unconditional Haar basis has MBP.

Even though the natural order on L^p (i.e., the pointwise order) is completely different from the order induced by the basis. Indeed, we have the following:

- **$\mathrm{L}^p(\Omega, \mu)$, where (Ω, μ) is a measurable space ($p \in [1, +\infty[$) endowed with the natural ordering cone $K := \{f \in \mathrm{L}^p(\Omega, \mu) : f \geqslant 0$ μ-a.e.$\}$ has MBP.**

Let (y_n) be a sequence in $\mathrm{L}^p(\Omega, \mu)$ converging to 0. There exists $(y'_n) \subset (y_n)$ and $y \in \mathrm{L}^p(\Omega, \mu)$ such that, μ-a.e. on Ω, $y'_n \to 0$ and, for all n, $|y'_n| \leqslant y$ [7]. Let us define $(\overline{y}_n)$ by $\overline{y}_n := \sup_{m \geqslant n} y'_m$ for all n. Clearly, $y'_n \leqslant \overline{y}_n \leqslant y$, $(\overline{y}_n)$ is non-increasing and, for a.e. $x \in \Omega$, $\lim \overline{y}_n(x) = \overline{\lim}\, y'_n(x) = 0$. By the Lebesgue dominated convergence theorem we have $\|\overline{y}_n\|_p \to 0$.

- **$\mathcal{C}_0(\Omega)$, where Ω is a non-empty subset of $\mathbb{R}^n$, endowed with the natural ordering cone has MBP.**

Indeed, if (y_n) is a sequence in $\mathcal{C}_0(\Omega)$ converging to 0, up to a subsequence, we can assume that $\sum_{n=1}^{\infty} \|y_n\|_\infty < \infty$. Let us define $(\overline{y}_n)$ by $\overline{y}_n = \sum_{m \geqslant n} |y_m|$ for all n. It is clear that $(\overline{y}_n)$ is a non-increasing sequence converging to 0 in $\mathcal{C}_0(\Omega)$ and $y_n \leqslant \overline{y}_n$ for all n.

Let us remark that, in general, we cannot hope to bound the original sequence (y_n) by a non-increasing sequence $(\overline{y}_n)$ converging to 0. We will see it for $Y = \mathrm{L}^p([0,1])$, $p \in [1, +\infty[$, K the natural ordering cone and the following sequence reminiscent of the Haar system: for $n \in \mathbb{N}_0$ and $k = 1, 2, 3, \ldots, 2^n$,

$$Y_n^{(k)}(x) := \begin{cases} 1 & \text{if } x \in \big[(k-1)/2^n, k/2^n\big] \\ 0 & \text{otherwise.} \end{cases}$$

and $Y_0^{(0)}(x) = 1$ for all $x \in [0,1]$. If we define the sequence (y_n) as $y_1 = Y_0^{(0)}$, $y_2 = Y_1^{(1)}$, $y_3 = Y_1^{(2)}, y_4 = Y_2^{(1)}, \ldots$ we can easily see that $y_n \to 0$ in $\mathrm{L}^p([0,1])$ and that we are forced to take $\overline{y}_n \geqslant \sup_{m \geqslant n} y_m$ in order to have $y_m \leqslant \overline{y}_m \leqslant \overline{y}_n$. But for any x in $[0,1]$ and for all n, $(\sup_{m \geqslant n} y_m)(x) = 1$ and thus $\overline{y}_n \not\to 0$ in $\mathrm{L}^p([0,1])$. However $(y'_n)_{n \geqslant 1} = \big(Y_n^{(1)}\big)_{n \geqslant 1}$ is a non-increasing subsequence of $(y_n)_{n \geqslant 1}$.

Proposition 7. *Let f be a function from X to Y where (Y, K) has MBP. If f is q-lsc then f is o-lsc.*

PROOF. Let (x_n) be a sequence in X converging to x_0 and for which there exists $(\varepsilon_n) \subset Y$ such that $\|\varepsilon_n\|_Y \to 0$ and $f(x_n) + \varepsilon_n \searrow$. Applying MBP to the sequence $(-\varepsilon_n)$, we get a subsequence $(\varepsilon_{n_k})_k \subset (\varepsilon_n)_n$ for which there exists a non-increasing sequence $(\overline{\varepsilon}_k) \subset Y$ converging to 0 in Y and such that $-\varepsilon_{n_k} \leqslant \overline{\varepsilon}_k$ for all k. Let k_0 be fixed.

For all $k \geqslant k_0$, we have:

$$\begin{aligned} f(x_{n_k}) &\leqslant f(x_{n_{k_0}}) + \varepsilon_{n_{k_0}} - \varepsilon_{n_k} \\ &\leqslant f(x_{n_{k_0}}) + \varepsilon_{n_{k_0}} + \bar{\varepsilon}_{k_0}. \end{aligned}$$

Since $x_{n_k} \to x_0$ and f is q-lsc at x_0, for all k_0:

$$f(x_0) \leqslant f(x_{n_{k_0}}) + \varepsilon_{n_{k_0}} + \bar{\varepsilon}_{k_0}.$$

As $\|\varepsilon_{n_k} + \bar{\varepsilon}_k\|_Y \to 0$, we have: $f(x_0) \leqslant f(x_{n_k}) + \mathrm{o}(1)$ and by Lemma 4, f is o-lsc at x_0. □

Without MBP, Proposition 7 is false. Let us take $Y = \mathbb{R}^2$ and $K = \{(x, 0) \in \mathbb{R}^2 : x \geqslant 0\}$. The function f defined from $\mathbb{R}$ to $\mathbb{R}^2$ by

$$f(x) := \begin{cases} (0,0) & \text{if } x = 0 \\ \big(|x|, |x| + 1\big) & \text{otherwise} \end{cases}$$

is q-lsc but not o-lsc at $x = 0$.

Let us now study the relation between quasi and order lower semi-continuity for bounded below functions.

Proposition 8. *Let f be a bounded below function from X to Y where the dimension of Y is finite. If f is q-lsc then f is o-lsc.*

PROOF. If $\operatorname{int} K \neq \varnothing$ then (Y, K) has MBP and by Proposition 7 we have the result. Let us suppose that $\operatorname{int} K = \varnothing$ and, without loss of generality, for all $x \in X : 0 \leqslant f(x)$, that is $f(X) \subset K \subset \operatorname{Aff} K$. Since, in the finite dimensional case, the interior of K for the topology relative to $\operatorname{Aff} K$ is non empty [21], the normed vector space $(\operatorname{Aff} K, K)$ has MBP and therefore we have: for all $(x_n) \subset X$ converging to x_0 for which there exists $(y_n) \subset \operatorname{Aff} K$ such that

$\|y_n\|_Y \to 0$ and $f(x_n) + y_n \searrow$, there exists $(g_n) \subset \operatorname{Aff} K$ such that $\|g_n\|_Y \to 0$ and $f(x_0) \leqslant f(x_n) + g_n$ for all n. We have to remove the restriction $(y_n) \subset \operatorname{Aff} K$. Suppose that $(y_n)_{n\geqslant 1} \subset Y$ is such that $\|y_n\|_Y \to 0$ and $f(x_n) + y_n \searrow$. Let us prove that $(y_n) \subset \operatorname{Aff} K$. For all n, let us write $y_n = a_n + t_n$ with $a_n \in \operatorname{Aff} K$, $t_n \in T$, where T is a topological complement of $\operatorname{Aff} K$ in Y, and $\|a_n\|_Y \to 0$, $\|t_n\|_Y \to 0$ as we can work with the equivalent norm $|||y_n||| := \|a_n\|_Y + \|t_n\|_Y$. From $f(x_n) + y_n \leqslant f(x_1) + y_1$, $n \geqslant 1$, we deduce

$$t_1 - t_n = -f(x_1) - a_1 + f(x_n) + a_n + k_n$$

for some $k_n \in K$. The fact that the right-hand side is an element of $\operatorname{Aff} K$ and $\operatorname{Aff} K \cap T = \{0\}$ imply $t_1 = t_n$ for all $n \geqslant 1$. Since $\|t_n\|_Y \to 0$, $t_n = 0$ for all n. □

Proposition 9. *Let f be a bounded below function from X to Y where (Y, K) is such that every order interval is compact. If f is o-lsc then f is q-lsc.*

PROOF. Let $b \in Y$ and (x_n) be a sequence in X converging to x_0 such that $f(x_n) \leqslant b$ for all n. Let a denote a lower bound of f. The interval $[a, b]$ is compact. Thus, one can extract a subsequence $(x'_n) \subset (x_n)$ such that $(f(x'_n))$ converges to some $y_0 \in [a, b]$. In other words, $f(x'_n) + \mathrm{o}(1) = y_0$ when $n \to \infty$. Since f is o-lsc at x_0, that implies $f(x_0) \leqslant f(x'_n) + \mathrm{o}(1) \leqslant b + \mathrm{o}(1)$ for all n. Since K is closed, we deduce $f(x_0) \leqslant b$ and that concludes the proof. □

Let us remark that every order interval of a finite dimensional Banach space partially ordered by a closed, convex, pointed cone is compact. Indeed, the only thing to see is that every order interval is bounded. Without loss of generality, one can assume that the interval is of the form $[0,b]$. If $\operatorname{int} K \neq \varnothing$, $\operatorname{int} K = \operatorname{cor} K \neq \varnothing$ and since K is algebraically closed and pointed, there exists a norm $\|\cdot\|$ on Y with the property that for all $b \in K : x \in [0,b] \Rightarrow \|x\| \leqslant \|b\|$ [17]. Then it follows immediatly, by equivalence of the norms in Y, that $[0,b]$ is bounded. If $\operatorname{int} K = \varnothing$, let us consider $\operatorname{Aff} K$ and T a topological complement of $\operatorname{Aff} K$ in Y. Since $\operatorname{int}_{\operatorname{Aff} K} K \neq \varnothing$, we have $\operatorname{cor}_{\operatorname{Aff} K} K = \operatorname{int}_{\operatorname{Aff} K} K \neq \varnothing$ and $K = \overline{K}^{\operatorname{Aff} K} = \operatorname{lin}_{\operatorname{Aff} K} K$. Then there exists a norm $\|\cdot\|_{\operatorname{Aff} K}$ on $\operatorname{Aff} K$ such that for all $x \in [0,b]$: $\|x\|_{\operatorname{Aff} K} \leqslant \|b\|_{\operatorname{Aff} K}$. If for all $y \in Y$, we define $|||y||| = \|a\|_{\operatorname{Aff} K} + \|t\|_Y$ where $y = a + t$ with $a \in \operatorname{Aff} K$, $t \in T$, $|||\cdot|||$ is an equivalent norm on Y such that for all $x \in [0,b]$: $|||x||| \leqslant |||b|||$ (since $[0,b] \subset K \subset \operatorname{Aff} K$). It follows:

Corollary 10. *Lef f be a bounded below function from X to Y where the dimension of Y is finite. Then f is q-lsc if and only if f is o-lsc.*

We now give some other examples of pairs (Y,K) for which the order intervals are compact.

If Y is a Banach space with unconditional basis and K is the natural cone associated to the basis, then the order intervals are compact.

We denote the unconditional normalized basis by $(e_i)_{i\geqslant 1}$ and the unconditional basis constant by C [3, 23]. Let $b = \sum_{i=1}^{\infty} b_i e_i \in Y$ and $(y_n) \subset [0,b]$. Since all the intervals $[0,b_i]$ of $\mathbb{R}$ are compact, by using a diagonal argument, one can extract a subsequence $(y'_n)_n \subset (y_n)_n$ such that all components converge: $\forall i \geqslant 1$, $y'_{ni} \xrightarrow[n\to\infty]{} y_i^\star \in [0,b_i]$. By unconditionality of the basis, $y^\star = \sum_{i=1}^{\infty} y_i^\star\, e_i$ belongs to Y. It remains to prove that (y'_n) converges to $y^\star$. Let $\varepsilon > 0$. Since the series $\sum_{i=1}^{\infty} b_i e_i$ converges in Y, there exists $I_\varepsilon \geqslant 1$ such that $\|\sum_{i=I_\varepsilon+1}^{\infty} b_i e_i\| < \varepsilon/4C$ and there exists $N_\varepsilon \geqslant 1$ such that for

$n \geqslant N_\varepsilon$, $\sum_{i=1}^{I_\varepsilon} |y'_{ni} - y^\star_i| \, \|e_i\| < \varepsilon/2$. So, by unconditionality of the basis, for $n \geqslant N_\varepsilon$:

$$\begin{aligned}\|y'_n - y^\star\| &\leqslant \varepsilon/2 + \Big\| \sum_{i=I_\varepsilon+1}^{\infty} (y'_{ni} - y^\star_i) e_i \Big\| \\ &\leqslant \varepsilon/2 + 2C \Big\| \sum_{i=I_\varepsilon+1}^{\infty} b_i e_i \Big\| \leqslant \varepsilon.\end{aligned}$$

Let us note that it is not true for $\mathrm{L}^\infty(\Omega,\mu)$ (resp. ℓ^∞) endowed with the natural order because the order interval $[-\chi_\Omega, \chi_\Omega]$ (resp. $[(-1,-1,\dots),(1,1,\dots)]$) correspond to the unit ball of $\mathrm{L}^\infty(\Omega,\mu)$ (resp. ℓ^∞).

Moreover, one can find bounded below o-lsc functions with values in ℓ^∞ which are not q-lsc . We can see it with the function f defined from $[0,1]$ to ℓ^∞ (endowed with the natural order) by:

$$f(x) = \begin{cases} (1,1,\dots) & \text{if } x = 0 \\ (\underbrace{0,\dots,0}_{n}, p_{1/2^n,1/2^{n+1}}(x), p_{1/2^{n+1},1/2^n}(x), 0, \dots) & \\ & \text{if } x \in \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right] \end{cases} \tag{3}$$

where $p_{a,b}$ is the affine map defined by

$$p_{a,b} : [a,b] \to [-1,0] : x \mapsto p_{a,b}(x) := -\frac{b-x}{b-a} \qquad (a,b \in \mathbb{R}).$$

This f is o-lsc at $x = 0$ (as it is impossible to find $(x_n) \subset [0,1]$, $x_n \to 0$ such that $f(x_n) + \mathrm{o}(1) \searrow$), not q-lsc at $x = 0$ ($\{f \leqslant 0\} =]0,1]$) and bounded below by $(-1,-1,\dots)$.

The same negative result holds for $\mathrm{L}^\infty(\mathbb{R})$. Let us consider the function f defined from $[0,1]$ to $\mathrm{L}^\infty(\mathbb{R})$ (endowed with the natural order) by:

$$f(x) = \begin{cases} \chi_\mathbb{R} & \text{if } x = 0 \\ -\chi_{]1/2^{n+1},1/2^n]} & \text{if } x \in \left]1/2^{n+1}, 1/2^n\right] \end{cases}$$

The function f is o-lsc at $x = 0$ (as it is impossible to find $(x_n) \subset [0,1]$, $x_n \to 0$ such that $f(x_n) + \mathrm{o}(1) \searrow$), f is not q-lsc at $x = 0$ ($x_n := 1/2^n \to 0$, $f(x_n) \leqslant 0$ but $f(0) = \chi_{\mathbb{R}} \nleqslant 0$) and f is bounded below in $\mathrm{L}^\infty(\mathbb{R})$ by $-\chi_{\mathbb{R}}$. Proposition 9 is thus false for $Y = \mathrm{L}^\infty$.

In $\mathcal{C}_0(\mathbb{R})$ the order intervals are not necessarily compact. Let us consider the order interval $[0, f]$ where f is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant -1 \text{ or } x \geqslant 2, \\ x+1 & \text{if } -1 \leqslant x \leqslant 0, \\ 1 & \text{if } 0 \leqslant x \leqslant 1, \\ 2-x & \text{if } 1 \leqslant x \leqslant 2. \end{cases}$$

It is impossible to extract a converging subsequence from the sequence $(f_n)_{n \geqslant 1} \subset [0, f]$ defined by

$$f_n(x) = \begin{cases} 0 & \text{if } x \leqslant 1/2^n \text{ or } x \geqslant 2, \\ 2^n x - 1 & \text{if } 1/2^n \leqslant x \leqslant 1/2^{n-1}, \\ 1 & \text{if } 1/2^{n-1} \leqslant x \leqslant 1, \\ 2-x & \text{if } 1 \leqslant x \leqslant 2, \end{cases}$$

because it is pointwise convergent to a discontinuous function. Let us take the function $g : [0,1] \to \mathcal{C}_0(\mathbb{R})$ defined by

$$g(t) = \begin{cases} h & \text{if } t = 0, \\ f_n & \text{if } t \in]1/2^{n+1}, 1/2^n], \end{cases}$$

where

$$h(x) = \begin{cases} 0 & \text{if } x \leqslant -1 \text{ or } x \geqslant 2, \\ x+1 & \text{if } -1 \leqslant x \leqslant 1/2, \\ 2-x & \text{if } 1/2 \leqslant x \leqslant 2. \end{cases}$$

This function g is o-lsc at $t = 0$ (since $f_n \nearrow_n$ and $\|f_n - f_m\|_{\mathcal{C}_0} = 1$ for all $n \neq m$, it is impossible to have $t_k \to 0$ with $g(t_k) + \mathrm{o}(1) \searrow$) but not q-lsc at $t = 0$ (because for all $t \neq 0$, $g(t) \leqslant f$ but $g(0) = h \nleqslant f$).

Proposition 9 is thus false for $Y = \mathcal{C}_0(\mathbb{R})$ endowed with the natural order.

In $\mathrm{L}^p(\mathbb{R}, \mu)$ (where $p \in [1, +\infty[$ and μ is the Lebesgue measure), the order intervals are neither necessarily compact. Let us consider the order interval $[0, \chi_{[0,1]}]$ and the sequence $(f_n)_{n \geqslant 1} \subset [0, \chi_{[0,1]}]$ defined by

$$f_n = \sum_{0 \leqslant i \leqslant 2^{n-1}-1} \chi_{[2i/2^n, (2i+1)/2^n]}.$$

We have $\|f_n - f_m\|_p^p = 1/2$ for all $n \neq m$ and thus it is impossible to extract a convergent subsequence of $(f_n)_{n \geqslant 1}$. Moreover, one can find a counter-example to Proposition 9 for $Y = \mathrm{L}^p(\mathbb{R}, \mu)$ endowed with the natural order. Let us consider the function $g : [0, 1] \to \mathrm{L}^p(\mathbb{R}, \mu)$ such that

$$g(t) = \begin{cases} 2\chi_{[0,1]} & \text{if } t = 0, \\ f_n & \text{if } t \in]1/2^{n+1}, 1/2^n]. \end{cases}$$

This function is o-lsc at $t = 0$ (because it is impossible to have $t_k \to 0$ and $g(t_k) + \mathrm{o}(1) \searrow$) but not q-lsc at $t = 0$ (since for all $t \neq 0$, $g(t) \leqslant \chi_{[0,1]}$ but $g(0) = 2\chi_{[0,1]} \nleqslant \chi_{[0,1]}$.

The following lemmas give an analogous for vector-valued functions of the well-known characterization of being lsc for real-valued functions, namely $f(x_0) \leqslant \underline{\lim}_{x \to x_0} f(x)$.

Lemma 11. *Let f be a function from X to Y, o-lsc at $x_0 \in X$. The following property holds:*

(4) *if $x_n \to x_0$ and $f(x_n) \to y_0$ then $f(x_0) \leqslant y_0$.*

PROOF. Let (x_n) be a sequence converging to x_0 in X. The fact that $(f(x_n))$ converges to some $y_0 \in Y$ can be written $f(x_n) + \mathrm{o}(1) = y_0$ when $n \to \infty$. Since f is o-lsc at x_0, that implies $f(x_0) \leqslant f(x_n) + \mathrm{o}(1) = y_0 + \mathrm{o}(1)$ and since K is closed, we have $f(x_0) \leqslant y_0$. □

Lemma 12. *Let f be a bounded below function from X to Y where (Y, K) has MBP and is such that every order interval is compact. If the property (4) holds then f is o-lsc at x_0.*

PROOF. Let (x_n) be a sequence converging to $x_0 \in X$ for which there exists $(\varepsilon_n) \subset Y$ converging to 0 and such that $f(x_n) + \varepsilon_n \searrow$. Since (Y, K) has MBP, there exists a subsequence $(\varepsilon_{n_k})_k \subset (\varepsilon_n)_n$ for which there exists a non-increasing sequence $(\overline{\varepsilon}_k) \subset Y$ converging to 0 and such that $-\varepsilon_{n_k} \leqslant \overline{\varepsilon}_k$ for all k. Let $a \in Y$ be such that $a \leqslant f(x)$ for all x in X and k_0 be fixed. For all $k \geqslant k_0$,

$$a \leqslant f(x_{n_k}) \leqslant f(x_{n_{k_0}}) + \varepsilon_{n_{k_0}} - \varepsilon_{n_k} \leqslant f(x_{n_{k_0}}) + \varepsilon_{n_{k_0}} + \overline{\varepsilon}_{k_0}.$$

Since $[a, f(x_{n_{k_0}}) + \varepsilon_{n_{k_0}} + \overline{\varepsilon}_{k_0}]$ is compact, there exists a subsequence $(x'_{n_k})_k \subset (x_{n_k})_{k=k_0}^{\infty}$ and $y_0 \in Y$ such that $f(x'_{n_k}) \xrightarrow[k\to\infty]{} y_0$. By property (4), $f(x_0) \leqslant y_0$. Thus $f(x_0) \leqslant y_0 = f(x'_{n_k}) + \mathrm{o}(1)$ which concludes the proof in virtue of Lemma 4. □

Let us mention that not every norm bounded function $f : X \to Y$ is bounded below, even if Y is finite dimensional. However, this implication is true when $\operatorname{int} K \neq \varnothing$.

It is readily proven using the compactness of closed balls, that when Y is finite dimensional and f is norm bounded then the property (4) implies that f is o-lsc at x_0. In fact more is true. Proceeding exactly like in the proof of Proposition 8, we obtain:

Lemma 13. *Let f be a bounded below function from X to Y where Y is finite dimensional. If property (4) holds then f is o-lsc at x_0.*

Let us remark that Lemma 13 is false if f is not bounded below. Let us take the function f defined from $[0, 1]$ to $(\mathbb{R}^2, \mathbb{R}^2_+)$ by:

$$f(x) = \begin{cases} (0,0) & \text{if } x = 0; \\ (1/n, 1) & \text{if } x \in \,]1/2^{n+1}, 1/2^n] \text{ and } n \text{ even}; \\ (-n, 1) & \text{if } x \in \,]1/2^{n+1}, 1/2^n] \text{ and } n \text{ odd}. \end{cases}$$

This function is not bounded below. Let $(x_n) \subset [0,1]$ be a sequence converging to 0: if $(f(x_n))$ converges it must necessarily be to $(0,1)$. Then $f(0) = (0,0) \leqslant (0,1) = \lim_{n\to\infty} f(x_n)$ and (4) is satisfied. But f is not o-lsc at 0. Indeed, take $x_n = 1/2^{2n+1}$ for all n: $(f(x_n))$ is non-increasing and it is impossible to have $f(0) \leqslant f(x_n) + \mathrm{o}(1)$.

Now, let us study the implication "o-lsc $\Rightarrow$ lsc" for order-bounded functions.

Proposition 14. *Let f be an order-bounded function from X to Y where (Y,K) is such that every order interval is compact. If f is o-lsc then f is lsc.*

PROOF. Let $\underline{b}$ and $\overline{b}$ in Y be such that $\underline{b} \leqslant f(x) \leqslant \overline{b}$ for all x in X. Since $[\underline{b}, \overline{b}]$ is compact, there exists a subsequence $(x'_n) \subset (x_n)$ and $y_0 \in [\underline{b}, \overline{b}]$ such that $f(x'_n) \xrightarrow[n\to\infty]{} y_0$. By Lemma 11, $f(x_0) \leqslant y_0 = f(x'_n) + \mathrm{o}(1)$ which concludes the proof in view of Lemma 4. $\square$

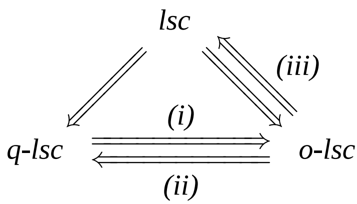
Proposition 14 is false if we do not have the compactness of the order intervals. Indeed, the function from $[0,1]$ to ℓ^∞ defined by (3) which is bounded below by $(-1,-1,-1,\dots)$ and above by $(1,1,\dots)$, is o-lsc but not q-lsc at 0 and thus not lsc at 0. It is neither possible to drop the assumption that f is bounded above as shown by considering $f : \mathbb{R} \to (\mathbb{R}^2, \mathbb{R}^2_+)$ defined by formula (1). Propositions 3 and 14 and Corollary 10, imply:

Corollary 15. *If Y is finite dimensional then the three notions of lower semi-continuity coincide for order bounded functions from X to Y.*

And by Propositions 3, 7, 9 and 14, we have:

Corollary 16. *If (Y, K) has MBP and is such that every order interval is compact, then the three notions of lower semi-continuity coincide for order bounded functions from X to Y.*

Summary. *Let $f : X \to (Y, K)$, $\dim Y > 1$. We have the following relationships:*



(i) Y *has MBP* *or* $\dim Y < \infty$ *and* f *is bounded below.*

(ii) f *is bounded below and order intervals are compact.*

(iii) f *is order bounded and order intervals are compact.*

We now study the vector-valued analogue of the following equivalence: "a real-valued function f is continuous if and only if it is lsc and usc." It is not the case for vector-valued functions. For example, let us take the function f defined from $\mathbb{R}$ to $\mathbb{R}^2$, endowed with the natural ordering cone, by

$$f(x) := \begin{cases} (0,-1) + (1,-1)/x & \text{if } x < 0 \\ (2,-2) & \text{if } x = 0 \\ (1,0) - (1,-1)/x & \text{if } x > 0 \end{cases}$$

This function is not continuous at $x = 0$ but it is q-lsc, q-usc and thus o-lsc and o-usc since $(\mathbb{R}^2, \mathbb{R}^2_+)$ has MBP.

Let us remark that this function is not bounded, neither below nor above. However,

Proposition 17. *Let f be an order bounded function from X to Y where (Y, K) is such that every order interval is compact. Then f is continuous iff f is o-lsc and o-usc.*

PROOF. It is obvious that continuity implies order lower and upper semi-continuity. Let (x_n) be a sequence in X converging to x_0 and $\underline{b}$, $\overline{b} \in Y$ be such that $\underline{b} \leqslant f(x) \leqslant \overline{b}$ for all x in X. One can extract a subsequence $(x'_n) \subset (x_n)$ such that $f(x'_n) \xrightarrow[n\to\infty]{} y_0 \in [\underline{b}, \overline{b}]$ and, by Lemma 11, if f is o-lsc and o-usc at x_0, then $f(x_0) \leqslant y_0$ and $f(x_0) \geqslant y_0$. Since K is pointed $f(x_0) = y_0$ and then $f(x'_n) \to f(x_0)$. An argument similar to the one in the proof of Lemma 4 concludes the proof. □

By Corollary 15, we have:

Corollary 18. *Let f be an order bounded function from X to Y where Y is finite dimensional. Then, whatever the notion of lower semi-continuity, f is continuous if and only if f is lower and upper semi-continuity.*

By Corollary 16, we have:

Corollary 19. *Let f be an order bounded function from X to Y where (Y, K) has MBP and is such that every order interval is compact. Then whatever the notion of lower semi-continuity, f is continuous if and only if f is lower and upper semi-continuous.*

Remark 20. In [26], this kind of result is considered for the notion of lower (upper) semi-continuity:

if (Y, K) is normal, f is continuous at x_0 iff f is lsc and usc at x_0.

Let us recall that (Y, K) is normal if the following property holds ([17, 19, 22]):

$$\exists \lambda > 0,\ \forall y \in K,\ \forall x, \quad x \in [0, y] \Rightarrow \|x\| \leqslant \lambda \|y\|.$$

For example, $\mathcal{C}_0(\Omega)$ and $\mathrm{L}^p(\Omega, \mu)$, $p \in [1, +\infty]$, endowed with the natural order are normal. Every Banach lattice (and in particular every Banach space with unconditional basis endowed with the natural cone) is normal. In virtue of the discussion page 13, (Y, K) is also normal whenever Y is finite dimensional.

Let us consider $Y = c_0$ and $x_n := \sum_{i=1}^{n} (-1)^{n+i} e_i$ where $(e_i)_{i=1}^{\infty}$ denotes the natural basis of c_0. Then (x_n) is a conditional basis of c_0 and the associated ordering cone is not normal (see [23, p. 477]).

3 Sum of lower semi-continuous functions

In [26], the authors prove that the sum of two lsc functions is lsc. That is no longer true for o-lsc functions. Let us consider the function s from $[0, 1]$ to ℓ^∞ (endowed with the natural order) defined by

$$s(x) = \begin{cases} (1, 1, \dots) & \text{if } x = 0 \\ (\underbrace{0, \dots, 0}_{n}, -p_{1/2^n, 1/2^{n+1}}(x), 1, 1, \dots) & \text{if } x \in \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right] \end{cases}$$

s is not o-lsc at $x = 0$ (since $(s(1/2^n))_n$ is non-increasing and it is impossible to have $s(0) \leqslant s(1/2^n) + \mathrm{o}(1)$). We can write $s = f + g$ where f is the o-lsc function defined by (3) and $g = s - f$ is o-lsc at $x = 0$ (and continuous elsewhere) because, for all x in $[0, 1]$, $g(0) = (0, 0, \dots) \leqslant g(x)$. Let us remark that the functions f and g are bounded below. However, we have the following facts.

Lemma 21. *The sum of a continuous function and an o-lsc function is o-lsc.*

PROOF. Let f and g be two functions defined from X to Y, f o-lsc at $x_0 \in X$ and g continuous at x_0. Let (x_n) be a sequence in X converging to x_0 and (ε_n) be a sequence in Y converging to 0 and such that $f(x_n) + g(x_n) + \varepsilon_n \searrow$. Since g is continuous at x_0, there exists $(g_n) \subset Y$ converging to 0 and such that, for all n, $g(x_0) = g(x_n) + g_n$. So, since $(f(x_n) + g(x_n) + \varepsilon_n)$ is non-increasing, we have for all n: $f(x_{n+1}) - g_{n+1} + \varepsilon_{n+1} \leqslant f(x_n) - g_n + \varepsilon_n$ with $\|-g_n + \varepsilon_n\|_Y \to 0$. By order lower semi-continuity of f at x_0, there exists (f_n) converging to 0 in Y and such that, for all n, $f(x_0) \leqslant f(x_n) + f_n$ and then $f(x_0) + g(x_0) \leqslant f(x_n) + g(x_n) + f_n + g_n$ with $\|f_n + g_n\|_Y \to 0$. This concludes the proof. □

Lemma 22. *If (Y, K) is such that every order interval is compact, then the sum of two o-lsc bounded below functions from X to Y is a q-lsc bounded below function. Moreover, the sum is o-lsc if we assume (Y, K) has MBP.*

The previous example shows that Lemma 22 is false without the compactness of order intervals.

PROOF. Let f and g be two o-lsc functions. We will prove $f + g$ is q-lsc. Let $b \in Y$ and (x_n) be a sequence in X converging to x_0 such that $f(x_n) + g(x_n) \leqslant b$. Since f and g are bounded below and the order intervals are compact, we have, up to a subsequence, $f(x_n) + \mathrm{o}(1) = y_0$ and $g(x_n) + \mathrm{o}(1) = z_0$ where $y_0, z_0 \in Y$. By order lower semi-continuity of f and g, we have $f(x_0) \leqslant f(x_n) + \mathrm{o}(1)$ and $g(x_0) \leqslant g(x_n) + \mathrm{o}(1)$. Thus $f(x_0) + g(x_0) \leqslant f(x_n) + g(x_n) + \mathrm{o}(1) \leqslant b + \mathrm{o}(1)$. Since K is closed, $f + g$ is q-lsc. By Proposition 7, if (Y, K) has MBP, the sum $f + g$ is o-lsc. □

Remark 23. In [9], the authors asserted that the sum of two q-lsc functions is q-lsc. This is not true in general. Indeed, let us take the functions f and g defined from $\mathbb{R}$ to $(\mathbb{R}^2, \mathbb{R}^2_+)$ by

$$f(x) = \begin{cases} (0,0) & \text{if } x = 0 \\ (-1,0) + (1,-1)/|x| & \text{if } x \neq 0 \end{cases}$$

$$g(x) = \begin{cases} (0,0) & \text{if } x = 0 \\ (-1,1)/|x| & \text{if } x \neq 0 \end{cases}$$

The function f (resp. g) is q-lsc at $x = 0$ since for all $b \in \mathbb{R}^2$ it is impossible to find (x_n) converging to 0 and such that $f(x_n) \leqslant b$ (resp. $g(x_n) \leqslant b$) for all n. But the function $f + g$ is clearly not q-lsc at $x = 0$. Let us point out that neither f nor g is bounded below.

For bounded below functions with values in a finite dimensional space Y, the assertion is correct in view of Lemma 22. On the other hand, if Y is infinite dimensional, it is not known whether this is still the case.

Concerning q-lsc functions, we can affirm:

Lemma 24. *Let f and g be two functions defined from X to Y where (Y, K) has MBP. If f is q-lsc and g is lsc then $f + g$ is q-lsc.*

PROOF. Let (x_n) be a sequence in X converging to x_0 and $b \in Y$ be such that $f(x_n) + g(x_n) \leqslant b$ for all n. Since g is lsc at x_0, by Lemma 2, there exists a sequence (g_n) converging to 0 in Y and such that $g(x_0) \leqslant g(x_n) + g_n$ for all n. So, for all n, $f(x_n) \leqslant b - g(x_0) + g_n$. Since (Y, K) has MBP, there exists a non-increasing sequence $(\overline{g}_n)$, converging to 0 in Y and such that, up to a subsequence, $g_n \leqslant \overline{g}_n$ for all n. Let n_0 be fixed. For $n \geqslant n_0$, we have: $f(x_n) \leqslant b - g(x_0) + \overline{g}_{n_0}$ and since f is q-lsc at x_0, $f(x_0) \leqslant b - g(x_0) + \overline{g}_{n_0}$. Then, since $\overline{g}_n \to 0$ and K is closed, $f(x_0) \leqslant b - g(x_0)$. This concludes the proof. □

4 Vector-valued Deville-Godefroy-Zizler perturbed minimization principle

Let f be a function from X to Y. The vector minimization problem under consideration is:

$$\text{find } \tilde{x} \in X \text{ such that } \{x \in X : f(x) \leqslant f(\tilde{x})\} = \{\tilde{x}\}.$$

Such $\tilde{x}$ is called an *efficient solution* of f and $f(\tilde{x})$ is said to be an *efficient point* of $f(X)$.

In general, for a given bounded below o-lsc function there is no reason why such a $\tilde{x}$ should exist in X. Our problem is to find a perturbation g as small as possible such that $(f+g)(X)$ admits an efficient point. We say that a function $f : X \to Y$ has a *strong efficient solution* on X at $\tilde{x}$ if $\tilde{x}$ is an efficient solution and $\|x_n - \tilde{x}\| \to 0$ whenever $(x_n) \subset X$ is such that $\|f(x_n) - f(\tilde{x})\| \to 0$ (every minimizing sequence is convergent).

Definition 25. Let C be a non-empty subset of Y and $\varepsilon \in \mathbb{R}_0^+$ be fixed. A point $y_0 \in C$ is said an *ε-infimal point of C in the direction of $e \in K \setminus \{0\}$* if and only if

$$\exists \rho > 0, \quad C \cap \big(y_0 - \varepsilon e + \mathrm{B}(0,\rho) - K\big) = \varnothing.$$

Let us mention that Ch. Tammer, in [24, 25], introduced the notion of ε-approximatively efficient points of $C \subset Y$: these are points y_0 satisfying $C \cap (y_0 - \varepsilon e - K \setminus \{0\}) = \varnothing$. This allows her to obtain an Ekeland-type variational principle by using a scalarization procedure for vector-valued q-lsc functions. The interior of the ordering cone is also required to be non-empty. Let us note that C. Finet [14] obtained the same result without any hypothesis on the interior of the ordering cone. The proof is also based on a scalarization procedure and follows the idea of I. Ekeland for the scalar case [12]. Lately, Ch. Tammer [16] obtained the same kind of results without any scalarization. The method used is similar to the one of Bishop and Phelps in their pioneer paper [4]. She introduced some notions of lower semi-continuity close to q-lsc and lsc. In Corollary 31, we obtain an Ekeland-type variational principle for o-lsc functions from our vector-valued Deville-Godefroy-Zizler perturbed minimization principle (Theorem 27).

Proposition 26 (Existence and localization of ε-infimal points). *If C is a non-empty bounded below subset of Y, then for every $\varepsilon \in \mathbb{R}_0^+$ and $e \in K \setminus \{0\}$ there exists y_0 an ε-infimal point of C in the direction of e. Moreover, given $\overline{y} \in C$ and $\delta \in \mathbb{R}_0^+$, one can assume that y_0 belongs to $C \cap \big(\overline{y} + \mathrm{B}(0,\delta) - K\big)$.*

PROOF. Let $\varepsilon \in \mathbb{R}_0^+$, $e \in K \setminus \{0\}$, $y_0 := \overline{y} \in C$ and $\delta \in \mathbb{R}_0^+$ be fixed. If y_0 is not an ε-infimal point of C in the direction of e, let $\rho_1 = \delta/2$ and get the existence of $y_1 \in C$ and $\xi_1 \in \mathrm{B}(0,\rho_1)$ such that $y_1 \leqslant y_0 - \varepsilon e + \xi_1$. If y_1 is an ε-infimal point of C in the direction of e, the proof is finished. If not, we repeat the same construction.

Let us define $\rho_n = \delta/2^n$ for all $n \in \mathbb{N}_0$. At the step $n \geqslant 1$, we have: $y_n \in C$ such that $y_n \leqslant y_0 - n\varepsilon e + \sum_{i=1}^n \xi_i$ with $\|\xi_i\|_Y < \rho_i$ for all $i \in \{1,\dots,n\}$. Let us suppose that for all $n \in \mathbb{N}_0$, y_n is not an ε-infimal point of C in the direction of e. Without loss of generality, we can suppose that C is bounded below by 0 in Y and then, for all $n \in \mathbb{N}_0$, $0 \leqslant y_n \leqslant y_0 - n\varepsilon e + \varphi_n$ with $\varphi_n = \sum_{i=1}^n \xi_i \in \mathrm{B}(0,\delta)$. Thus

$$e \in \bigcap_{n=1}^{\infty} \Big(\mathrm{B}\big(0, \frac{r}{n}\big) - K\Big), \qquad \text{where } r := \frac{\|y_0\|_Y + \delta}{\varepsilon}.$$

Since K is closed, this intersection is equal to $-K$ and since K is pointed this implies that $e = 0$, a contradiction. $\square$

We call a bump function on X a real-valued function on X with bounded non-empty support. For example, the function b defined on X by

$$b(x) = \begin{cases} 1 - \|x\| & \text{if } \|x\| \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

is a continuous bump on X.

Let us denote, for a bounded function $f : X \to Y$, $\|f\|_\infty = \sup_{x \in X} \|f(x)\|_Y$.

Theorem 27. *Let $(Z, \|\cdot\|_Z)$ be a complete convex cone of norm bounded, bounded below, continuous functions from X to Y such that:*

(i) for all $g \in Z$, $\|g\|_\infty \leqslant \|g\|_Z$,

(ii) Z is translation invariant, i.e. if $g \in Z$ and $x \in X$ then $\tau_x g : X \to Y$ given by $\tau_x g(t) := g(t - x)$ is in Z and $\|\tau_x g\|_Z = \|g\|_Z$.

(iii) Z is dilation invariant, i.e. if $g \in Z$ and $\alpha \in \mathbb{R}$ then $g^\alpha : X \to Y$ given by $g^\alpha(t) := g(\alpha t)$ is in Z.

(iv) there exists a continuous and norm bounded bump function $b : X \to \mathbb{R}$ and an element $e \in K \setminus \{0\}$ such that $b(0) > 0$ and $\tilde{b} = -be$ belongs to Z.

Let $f : X \to Y$ be an o-lsc bounded below function. Then the set of all $g \in Z$ such that $f + g$ admits a strong efficient solution is dense in Z.

Examples 28. Let us first give some examples of complete convex cones $(Z, \|\cdot\|_Z)$ of norm bounded, bounded below, continuous functions from X to Y satisfying the conditions (i)–(iv). An easy way to construct such a cone of perturbations is to take $(Z, \|\cdot\|_Z) = (\tilde{Z}e, \|\cdot\|_{\tilde{Z}})$ where $e \in K$ and $\tilde{Z}$ is a complete convex cone of *real-valued* continuous and bounded functions defined on X and such that $(\tilde{Z}, \|\cdot\|_{\tilde{Z}})$ satisfies (i)–(iv) of the scalar version of Theorem 27. Examples of cones $\tilde{Z}$ are given in [11, 20].

Let $\tilde{Z} = L$ be the space of all bounded real-valued Lipschitz continuous functions g on X with $\|g\|_L = \|g\|_\infty + \|g\|_{\mathrm{Lip}}$ where

$$\|g\|_{\mathrm{Lip}} := \sup\Big\{\frac{|g(x) - g(y)|}{\|x - y\|} : x, y \in X,\ x \neq y\Big\}.$$

It is straightforward to prove that L is a Banach space which satisfies hypotheses (i)–(iii). Concerning hypothesis (iv), one can apply the construction exposed in [11, 20] to produce a bounded Lipschitzian bump function.

Let us recall that a bornology on X, denoted by β, is any family of bounded sets whose union is all X, which is closed under reflection through the origin (that is $S \in \beta$ implies $-S \in \beta$), under multiplication by positive scalars and is directed upwards (that is the union of any two members of β is contained in some member of β). There are many possibilities. Let us describe the smallest and the largest ones : the *Gâteaux* bornology $\beta = G$ consisting of all finite symmetric sets and the *Fréchet* bornology $\beta = F$ consisting of all bounded symmetric sets. A function $f : X \to Y$ is said to be β*-differentiable* at x and $T \in \mathcal{L}(X, Y)$ is called its β*-derivative* at x, if for each $S \in \beta$,

$$\lim_{t \xrightarrow{>} 0} \frac{f(x + ty) - f(x)}{t} = T(x) \quad \text{uniformly for } y \in S.$$

We denote the β-derivative of f at x by $\partial_\beta f(x)$. It is clear that we find again the well-known Gâteaux (resp. Fréchet) derivative with $\beta = G$

(resp. $\beta = F$). We can take for $\tilde{Z}$ the Banach space D_β of all real-valued functions defined on X that are bounded, Lipschitz continuous and β-differentiable equipped with the norm

$$\|g\|_{D_\beta} := \|g\|_\infty + \|\partial_\beta g\|_\infty$$

(cfr. [20] for a proof that this space is complete and verifies hypotheses (i)–(iv) in the scalar case).

We must pay attention to the fact that the cone Z of *vector-valued* functions which are *bounded below*, norm bounded, Lipschitz continuous (resp. and β-differentiable) equipped with the norm

$$\|g\|_Z := \|g\|_\infty + \sup\Big\{\frac{\|g(x) - g(y)\|_Y}{\|x - y\|_X} : x, y \in X,\ x \neq y\Big\}$$

(resp. $\|g\|_Z := \|g\|_{D_\beta}$), is generally not complete. Actually, it is true when the interior of the ordering cone is non-empty (since in this case every norm bounded function is bounded below). Let us mention that the normality of the ordering cone (usually a good notion to link topology and order) is not useful here since norm bounded sets need not be order bounded even if the cone is normal. (Let us mention that the converse is true, cfr. [19].)

Remark 29. C. Finet [14] proved this theorem for a *lsc* bounded below function f with a complete convex cone Z of norm bounded, bounded below, *lsc* perturbations. She proceeds by scalarization, using the fact that if $y^\star \in K^\star := \{y^\star \in Y^\star : \forall y \in K,\ y^\star(y) \geqslant 0\}$ and f is lsc then $y^\star \circ f$ is lsc [26]. This fact is not true for q-lsc functions. Indeed, let us take the function f from $\mathbb{R}$ to $\mathbb{R}^2$ (endowed with the natural order) defined by (2) and $y^\star : \mathbb{R}^2 \to \mathbb{R}$ be such that $y^\star(x, y) = x + y$; f is q-lsc and q-usc at $x = 0$ but

$$(y^\star \circ f)(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

and then $y^\star \circ f$ is neither lsc nor usc.

What is the situation for a q-lsc function f perturbed by q-lsc functions? In general we don't know. Following the proof of [9], we get the same result when the dimension of Y is finite or when (Y, K) has compact order intervals (see Lemma 22). In Theorem 27, we perturb the o-lsc function f with continuous functions (see Lemma 21). If we assume that (Y, K) has MBP and has compact order intervals then we got the theorem with o-lsc perturbations instead of continuous perturbations (see Lemma 22).

We prove this theorem without scalarization process.

PROOF. Let $\varepsilon > 0$ be fixed. We want to prove that there exists $g \in Z$, $\|g\|_Z \leqslant \varepsilon$, and $\tilde{x} \in X$ such that $\tilde{x}$ is a strong efficient solution of $f + g$.

(i) We can suppose that $b(0) = 1$ and $\|e\|_Y = 1$. Moreover, there exists some $r > 0$ such that $b(x) = 0$ whenever $\|x\| \geqslant r$. Let us define $b_1 : X \to \mathbb{R}$ by $b_1(x) = b(2rx)$ whose support is now in $\mathrm{B}_X(0, 1)$. Since $f(X) \subset Y$ is bounded below, there exists $x_1 \in X$ such that $f(x_1)$ is an $\varepsilon_1 := \varepsilon/(2\|b_1 e\|_Z)$-infimal point of $f(X)$ in the direction e, that is:

(5) $\exists \rho_1 > 0, \ \forall x \in X, \ \forall \xi \in \mathrm{B}_Y(0, \rho_1), \quad f(x) \nleqslant f(x_1) - \varepsilon_1 e + \xi.$

Let us define $g_1 : X \to Y$ by $g_1(x) := -b_1(x - x_1)\varepsilon_1 e$ so that $g_1(x_1) = -\varepsilon_1 e$. We have $g_1 \in Z$ and $\|g_1\|_Z \leqslant \varepsilon/2$. If we set

$$A_1 := \big\{x \in X : \exists \xi \in \mathrm{B}_Y(0, \rho_1), \\ (f + g_1)(x) \leqslant (f + g_1)(x_1) + \xi\big\},$$

then $A_1 \subset \operatorname{supp} g_1 \subset \mathrm{B}_X(x_1, 1)$, because if $x \notin \operatorname{supp} g_1$, $g_1(x) = 0$ and, by (5), $x \notin A_1$. If $A_1 = \{x_1\}$, then x_1 is an efficient solution of

$f + g_1$. Moreover, this efficient point is strong. Indeed, let (u_n) be a sequence in X such that $(f + g_1)(u_n) \to (f + g_1)(x_1)$. Then, there exists $(v_n) \subset Y$ converging to 0 such that, for all n, $(f + g_1)(u_n) = (f + g_1)(x_1) + v_n$. For n large enough, $\|v_n\|_Y \leqslant \rho_1$. This implies that $u_n \in A_1 = \{x_1\}$ for n large enough and consequently $u_n \to x_1$. Let us suppose that $A_1 \neq \{x_1\}$.

(ii) Let us define $b_2 : X \to \mathbb{R}$ by $b_2(x) := b_1(2x)$ whose support lies in $\mathrm{B}_X(0, 1/2)$. Since $(f + g_1)(X)$ is bounded below in Y, by Proposition 26 applied with $\varepsilon_2 := \min\{\varepsilon/2^2, \rho_1/4\}/\|b_2 e\|_Z$, $\overline{y} := (f + g_1)(x_1)$ and $\delta := \rho_1/4$, there exists $x_2 \in X$ such that $f(x_2)$ is an ε_2-infimal point of $(f + g_1)(X)$ in the direction of e and belonging to $(f + g_1)(X) \cap \big((f + g_1)(x_1) + \mathrm{B}_Y(0, \delta) - K\big)$, that is:

$$\exists y_2 \in \mathrm{B}_Y(0, \rho_1/4), \quad (f + g_1)(x_2) \leqslant (f + g_1)(x_1) + y_2$$
$$\exists \rho_2 > 0,\ \forall x \in X,\ \forall \xi \in \mathrm{B}_Y(0, \rho_2),$$
$$(f + g_1)(x) \nleqslant (f + g_1)(x_2) - \varepsilon_2 e + \xi$$

Without loss of generality, we can assume that $\rho_2 \leqslant \rho_1/4$. Let us define $g_2 : X \to Y$ by $g_2(x) := -b_2(x - x_2)\varepsilon_2 e$. We have $g_2 \in Z$ and $\|g_2\|_Z \leqslant \min\{\varepsilon/2^2, \rho_1/4\}$. If we set

$$A_2 := \big\{x \in X : \exists \xi \in \mathrm{B}_Y(0, \rho_2),$$
$$(f + g_1 + g_2)(x) \leqslant (f + g_1 + g_2)(x_2) + \xi\big\},$$

then $A_2 \subset \operatorname{supp} g_2 \subset \mathrm{B}_X(x_2, 1/2^2)$. If $A_2 = \{x_2\}$, then x_2 is a strong efficient solution of $f + g_1 + g_2$. Let us suppose that $A_2 \neq \{x_2\}$.

(iii) Let us suppose we have carried out the construction until step $n - 1$ and let us perform the step n. Let us write $\overline{g}_{n-1} := \sum_{k=1}^{n-1} g_k$ and $b_n := b_1(2^n \cdot)$ so that $\operatorname{supp} b_n \subset \mathrm{B}_X(0, 1/2^n)$. Proposition 26 applied to $(f + \overline{g}_{n-1})(X)$ with $\varepsilon_n := \min\{\varepsilon/2^n, \rho_{n-1}/4\}/\|b_n e\|_Z$,

$\overline{y} := (f + \overline{g}_{n-1})(x_{n-1})$ and $\delta := \rho_{n-1}/4$ gives the existence of $x_n \in X$, $\rho_n > 0$ and $y_n \in \mathrm{B}_Y(0, \rho_{n-1}/4)$ such that

$$(f + \overline{g}_{n-1})(x_n) \leqslant (f + \overline{g}_{n-1})(x_{n-1}) + y_n, \tag{6}$$

$$\forall x \in X,\ \forall \xi \in \mathrm{B}_Y(0, \rho_n), \quad (f + \overline{g}_{n-1})(x) \nleqslant (f + \overline{g}_{n-1})(x_n) - \varepsilon_n e + \xi. \tag{7}$$

Without loss of generality, we can assume $\rho_n \leqslant \rho_{n-1}/4$. Let us define $g_n : X \to \mathbb{R}$ by $g_n(x) = -b_n(x - x_n)\varepsilon_n e$. It is easy to check $g_n \in Z$ and $\|g_n\|_Z \leqslant \min\{\varepsilon/2^n, \rho_{n-1}/4\}$. If we set

$$A_n := \big\{x \in X : \exists \xi \in \mathrm{B}_Y(0, \rho_n), \quad (f + \overline{g}_n)(x) \leqslant (f + \overline{g}_n)(x_n) + \xi\big\},$$

then $A_n \subset \operatorname{supp} g_n \subset \mathrm{B}_X(x_n, 1/2^n)$. If $A_n = \{x_n\}$, then x_n is a strong efficient solution of $f + \overline{g}_n$.

(iv) Let us suppose that, for all n, $A_n \neq \{x_n\}$. Since $\|y_{n+1}\| < \rho_n$, we have that $x_{n+1} \in A_n \subset \mathrm{B}_X(x_n, 1/2^n)$ and thus (x_n) is a Cauchy sequence in X. So, there exists some $\tilde{x} \in X$ such that $x_n \to \tilde{x}$. Also $\|g_n\|_Z \leqslant \varepsilon/2^n$ for all n implies that there exists some $g \in Z$ such that $\overline{g}_n \xrightarrow{Z} g$. So $g = \overline{g}_n + h_n$ with $\|g\|_Z \leqslant \varepsilon$ and $h_n = \sum_{i>n} g_i$.

(v) We want to show that $\tilde{x}$ is an efficient solution of $f + g$. In view of (6) and because $g_{n+1}(x_{n+1}) = -\varepsilon_{n+1} e \leqslant 0$, we have

$$\begin{aligned}(f + \overline{g}_{n+1})(x_{n+1}) &\leqslant (f + \overline{g}_n)(x_{n+1}) \\ &\leqslant (f + \overline{g}_n)(x_n) + y_{n+1}\end{aligned} \tag{8}$$

for all n and therefore, adding $\sum_{i>n+1} y_i$ to both sides, $(f+g)(x_n) + \big(-h_n(x_n) + \sum_{i>n} y_i\big) \underset{n}{\searrow}$ with $\|-h_n(x_n) + \sum_{i>n} y_i\|_Y \to 0$. Since g is continuous, by Lemma 21, $f + g$ is o-lsc at $\tilde{x}$ and thus

$$(f + g)(\tilde{x}) \leqslant (f + g)(x_n) + v_n \quad \text{with } \|v_n\|_Y \to 0. \tag{9}$$

Let now $x \in X$ be such that $(f+g)(x) \leqslant (f+g)(\tilde{x})$. We have to prove that $x = \tilde{x}$. From (9) we deduce

$$(f+g)(x) \leqslant (f+g)(x_n) + v_n \quad \text{for all } n. \tag{10}$$

Let $n_0 \in \mathbb{N}_0$ be fixed. By (10), we have:

$$(f+\overline{g}_{n_0})(x) \leqslant (f+\overline{g}_n)(x_n) + h_n(x_n) + v_n - h_{n_0}(x)$$

and, by using (8) repeatedly, we get, for any $n \geqslant n_0$,

$$\begin{aligned}(f+\overline{g}_{n_0})(x) \leqslant (f+\overline{g}_{n_0})&(x_{n_0}) \\ &+ \sum_{n_0<i\leqslant n} y_i + h_n(x_n) + v_n - h_{n_0}(x).\end{aligned}$$

We will show that, when n is large,

$$\Big\| \sum_{n_0<i\leqslant n} y_i + h_n(x_n) + v_n - h_{n_0}(x) \Big\|_Y < \rho_{n_0}. \tag{11}$$

Then $x \in A_{n_0} \subset \mathrm{B}_X(x_{n_0}, 1/2^{n_0})$. Since n_0 is arbitrary, $x = \tilde{x}$. Formula (11) results from the following three estimates:

- $\|y_i\|_Y \leqslant \rho_{i-1}/4 \leqslant \rho_{i-2}/4^2 \leqslant \cdots \leqslant \rho_{n_0}/4^{i-n_0}$ for all $i > n_0$. So, for any $n \geqslant n_0$,

$$\Big\| \sum_{n_0<i\leqslant n} y_i \Big\|_Y \leqslant \sum_{i=n_0+1}^{\infty} \|y_i\|_Y \leqslant \sum_{i\geqslant 1} \rho_{n_0}/4^i = \rho_{n_0}/3;$$

- $\|h_n(x_n) + v_n\|_Y \leqslant \|h_n\|_\infty + \|v_n\|_Y \leqslant \|h_n\|_Z + \|v_n\|_Y < \rho_{n_0}/3$ for n large enough;
- $\|g_i\|_Z \leqslant \rho_{i-1}/4$, so

$$\begin{aligned}\|h_{n_0}(x)\|_Y \leqslant \|h_{n_0}\|_\infty \leqslant \|h_{n_0}\|_Z &\leqslant \sum_{i>n_0} \|g_i\|_Z \\ &\leqslant \sum_{i>n_0} \rho_{i-1}/4 \leqslant \sum_{i>n_0} \rho_{n_0}/4^{i-n_0} = \rho_{n_0}/3.\end{aligned}$$

(vi) Finally, let us prove that $\tilde{x}$ is a strong efficient solution of $f+g$. Let (u_m) be a sequence such that $(f+g)(u_m) \to (f+g)(\tilde{x})$. Let $n_0 \in \mathbb{N}_0$ be fixed. When m is large enough, one can write

$$(f+g)(u_m) \leqslant (f+g)(\tilde{x}) + w_m \tag{12}$$

with $\|w_m\|_Y \leqslant \rho_{n_0}/6$. Using (9) and (8), one deduces as before that, for all $n \geqslant n_0$,

$$\begin{aligned}(f+\overline{g}_{n_0})(u_m) \leqslant (f+\overline{g}_{n_0})(x_{n_0})\\ + \sum_{n_0<i\leqslant n} y_i + h_n(x_n) + v_n + w_m - h_{n_0}(u_m).\end{aligned}$$

Similar estimates to the ones above hold:

- $\left\|\sum_{n_0<i\leqslant n} y_i\right\|_Y \leqslant \rho_{n_0}/3$ for all $n > n_0$;
- $\|h_n(x_n) + v_n\|_Y < \rho_{n_0}/6$ for n large enough;
- $\|h_{n_0}(u_m)\|_Y \leqslant \rho_{n_0}/3$ for all m.

As a consequence,

$$(f+\overline{g}_{n_0})(u_m) \leqslant (f+\overline{g}_{n_0})(x_{n_0}) + \xi \qquad \text{where } \|\xi\|_Y < \rho_{n_0}.$$

Summing up, we have shown that $\forall n_0, \exists m_0, \forall m \geqslant m_0, u_m \in A_{n_0} \subset \mathrm{B}_X(x_{n_0}, 1/2^{n_0})$. From this, one easily concludes that $u_m \xrightarrow[m]{} \tilde{x}$. □

Remark 30. Let us note that, because the last part of the proof relies only on (12), we have in fact proven that

$$\forall (x_n) \subset X, \qquad \varphi(x_n) \leqslant \varphi(\tilde{x}) + \mathrm{o}(1) \quad \Rightarrow \quad x_n \to \tilde{x}. \tag{13}$$

for $\varphi := f+g$. Clearly, any point $\tilde{x}$ satisfying (13) is a strong efficient point. For real-valued bounded below functions φ, (13) is equivalent to the fact that $\tilde{x}$ is strongly efficient. This however is no longer true for vector-valued functions φ as the example below will show. We believe that (13) is a notion of strong efficiency better suited for vector valued functions.

Let $\varphi : \mathbb{R} \to \ell^\infty$ be the continuous extension of the function

$$\mathbb{Z} \to \ell^\infty : n \mapsto \begin{cases} (0, 0, \dots) & \text{if } n = 0, \\ \frac{1}{|n|} e_{|n|} - e_{|n|+1} & \text{if } n \neq 0, \end{cases}$$

(where e_n is the vector whose all components are 0 except the n^{th} one which is equal to 1) obtained by joining consecutive points by linear interpolation. The reader will easily check that $x = 0$ is a strong efficient point of φ but does not satisfy (13) (indeed $x_n := n$ is such that $\varphi(x_n) \leqslant \varphi(0) + e_n/n$ but $x_n \not\to 0$).

We now give some applications of Theorem 27.

According to Examples 28, we can take $(Z, \|\cdot\|_Z) = (Le, \|\cdot\|_L)$ from which we deduce the following vector-valued version of the Ekeland variational principle.

Corollary 31. *Let $f : X \to (Y, K)$ be an o-lsc bounded below function. For every $\varepsilon \in \mathbb{R}_0^+$ and every $e \in K$ there exists $x_{\varepsilon,e} \in X$ both an ε-approximatively efficient point of f in the direction of e and a strong efficient solution of $f_{\varepsilon,e} := f + \varepsilon\|x_{\varepsilon,e} - \cdot\|e$.*

PROOF. Let $\varepsilon > 0$ and $e \in K$ be fixed. By Theorem 27, there exists $g = \tilde{g}e \in Z = Le$ and $x_{\varepsilon,e} \in X$ such that $x_{\varepsilon,e}$ is a strong efficient solution of $f + g$, that is :

(14) for all x in $X \setminus \{x_{\varepsilon,e}\}$, $\quad f(x) \not\leqslant f(x_{\varepsilon,e}) + g(x_{\varepsilon,e}) - g(x)$.

Since, for all $x \in X$, $g(x_{\varepsilon,e}) - g(x) = \big(\tilde{g}(x_{\varepsilon,e}) - \tilde{g}(x)\big)e \geqslant -\varepsilon e$, we have, by transitivity of the order, $f(x) \nleqslant f(x_{\varepsilon,e}) - \varepsilon e$ for all x in X.

So, $x_{\varepsilon,e}$ is an approximatively efficient point of f in the direction of e.

On the other hand, using for all x in $X \setminus \{x_{\varepsilon,e}\}$: $g(x) - g(x_{\varepsilon,e}) \leqslant \varepsilon\|x_{\varepsilon,e} - x\|e$ and (14), we deduce that $f(x) \nleqslant f(x_{\varepsilon,e}) - \varepsilon\|x_{\varepsilon,e} - x\|e$ for all x in $X \setminus \{x_{\varepsilon,e}\}$.

Thus $\{x_{\varepsilon,e}\} = \big\{x \in X : f(x) + \varepsilon\|x_{\varepsilon,e} - x\|e \leqslant f(x_{\varepsilon,e})\big\}$ and then $x_{\varepsilon,e}$ is an efficient solution of $f_{\varepsilon,e} := f + \varepsilon\|x_{\varepsilon,e} - \cdot\,\|e$.

Let us prove now that this solution is a strong solution. Let us suppose that the sequence $(x_n)_{n\geq 1} \subset X$ is such that $f(x_n) + \varepsilon\|x_{\varepsilon,e} - x_n\| \leqslant f(x_0) + \mathrm{o}(1)$. We have

$$\begin{aligned} f(x_n) + g(x_n) - g(x_{\varepsilon,e}) &\leqslant f(x_n) + \varepsilon\|x_{\varepsilon,e} - x_n\|e \\ &\leqslant f(x_{\varepsilon,e}) + \mathrm{o}(1). \end{aligned}$$

Thus $(f+g)(x_n) \leqslant (f+g)(x_{\varepsilon,e}) + \mathrm{o}(1)$. Since $x_{\varepsilon,e}$ is a strong efficient solution of $f+g$, $x_n \xrightarrow[n\to\infty]{} x_{\varepsilon,e}$ which proves that $x_{\varepsilon,e}$ is a strong efficient solution of $f_{\varepsilon,e}$. $\square$

On the other hand, in view of Examples 28, we can also consider $(Z, \|\cdot\|_Z) = (D_\beta e, \|\cdot\|_{D_\beta})$ and then get a vector-valued version of the Borwein-Preiss smooth perturbed minimization principle.

Corollary 32 (Smooth perturbed minimization principle). *Let X be a Banach space that admits a Lipschitz continuous bump function which is β-differentiable. Then for every $f : X \to Y$ o-lsc, bounded below, and for every $\varepsilon > 0$, there exists a function $g : X \to Y$ which is Lipschitz continuous and β-differentiable such that $\|g\|_\infty \leqslant \varepsilon$, $\|\partial_\beta g\|_\infty \leqslant \varepsilon$ and $f+g$ admits a strong efficient solution.*

Remark 33. Let us note that order lower semi-continuity is not necessary in order to obtain a Deville-Godefroy-Zizler principle. Indeed, consider for example the real-valued upper semi-continuous function f defined on a Banach space X by $f(x) := \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}$ and the space L of all bounded real-valued Lipschitz continuous functions g on X equipped with the norm $\|g\|_L = \|g\|_\infty + \|g\|_{\text{Lip}}$ for cone of perturbations.

On the other hand, Theorem 27 does not hold for any function f. Let us consider the function $f : [0,1] \to \mathbb{R}$ introduced in [1]:

$$f(x) := \begin{cases} (-1)^q(1 - 1/q) & \text{if } x = p/q,\ (p,q) = 1,\ x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

This function is bounded below, not lsc, and for all continuous function $g : [0,1] \to \mathbb{R}$, the perturbed function $f + g$ admits no minimum.

The same facts hold in the vector-valued case as the non o-lsc function $h : [0,1] \to (\mathbb{R}^2, \mathbb{R}^2_+)$ defined by $h(x) := \big(f(x), f(x)\big)$ shows.

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