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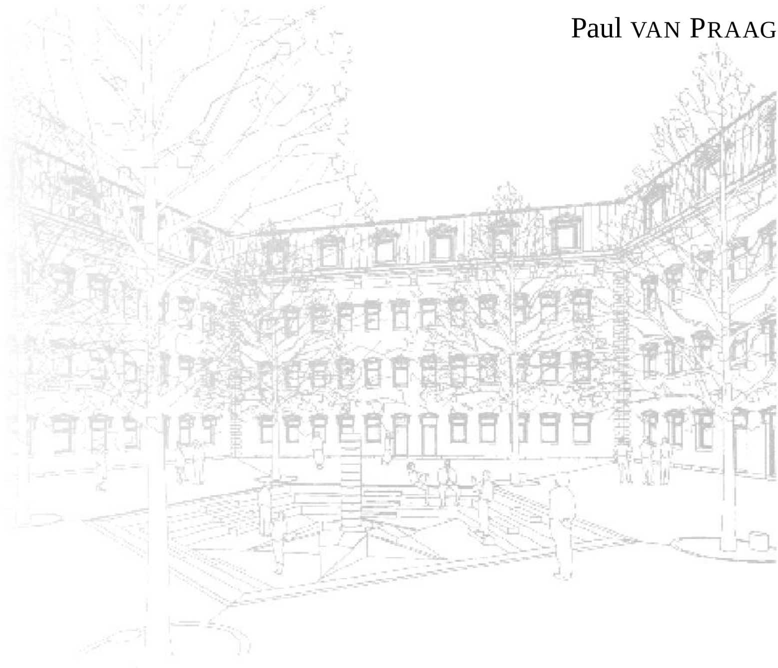

Institut de Mathématique et d'Informatique
Université de Mons-Hainaut
Phone: +32 65 37 35 07 — Fax: +32 65 37 33 18
Web: http://www.umh.ac.be/math/institut

UMH
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Dedicated to Martine Oppitz whose first words to the author of this paper were about quaternions.

1 Terminology, notation and aims of the paper

1.1 Throughout this text a *skew field*, or a *sfield*, is a ring with a unit element 1, in which every non zero element has an inverse.

Given D an sfield with centre F and σ an *involution* of D, i.e. a transformation $x \mapsto \sigma(x)$ of D for which $\sigma(x+y) = \sigma(x) + \sigma(y)$, $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma^2(x) = x$, for any $x, y \in D$.

Jean DIEUDONNÉ, in his work on the structure of unitary groups, says in [1, p. 72] that D is a *reflexive* sfield if all the elements of F are fixed by σ and if $x + \sigma(x)$ and $x\sigma(x)$ belong to F, for any $x \in D$. He shows (by completing his proof in [2]) that non commutative reflexive sfields are generalized quaternions sfields. His proof, amongst other things, relies on theorems concerning the structure of sfields of finite rank over their center, proved earlier this century in the '20 s. Now, Eliakim Hastings MOORE had already stated and provided elementary proof for the above-mentioned result provided by DIEUDONNÉ for sfields of characteristic other than 2. In 1922, MOORE has defined a determinant for hermitian matrices with coefficients in those fields whose elements, fixed by the involution, are central [3]. For MOORE's motivations, see the introduction of [4], written after his death, for the American Mathematical Society; for biographic information see [5]. MOORE's proof [4, pp. 104–107] is based on an identity valid for any sfield with involution whose elements fixed by the involution are central. This identity, which was expressed by HAMILTON himself [6, p. 317] for usual quaternions, is, in this case a translation of the identity

$$\langle \vec{z} \times \vec{t}, \vec{y} \rangle \vec{x} - \langle \vec{z} \times \vec{t}, \vec{x} \rangle \vec{y} - \langle \vec{x} \times \vec{y}, \vec{t} \rangle \vec{z} + \langle \vec{x} \times \vec{y}, \vec{z} \rangle \vec{t} = 0, \quad (1)$$

in which x, y, z and t are vectors in usual three-dimensional space, and $\langle \vec{x}, \vec{y} \rangle$ and $\vec{x} \times \vec{y}$ are respectively the scalar and vector products of x and y. Here we sketch out DIEUDONNÉ's non elementary proof, present MOORE's proof because the book [4] is difficult to obtain, and also present another elementary proof based on the fact that if $V = \{x \in D \mid \sigma(x) = -x\}$, then V is a vector space on the subfield F_0 of the elements of F fixed by σ and the application $x \mapsto x^2$ is a regular quadratic form on that vector space.

1.2 Throughout this text, unless explicitely stated otherwise, F is a commutative sfield with characteristic unequal to 2, D is sfield with centre F and σ is an involution of D.

Example 1. D is a commutative sfield $(D = F)$ and σ is the identity map.

Example 2. D is a commutative sfield, a quadratic extension of a commutative sfield F_0: $D = F_0(a)$, where $a^2 \in F_0$. The elements of D are written $a_0 + a_1\alpha$ (e.g., see [7]) and

$$\sigma(a_0 + \alpha_1 a) = a_0 - \alpha_1 a.$$

Example 3. **a)** D is a sfield of generalized quaternions on F, or more simply a sfield of quaternions on F, constructed as follows. Given α, β belonging to a commutative sfield, W a vector space of dimension 4 on F, and $\{e, i, j, k\}$ a base of W, it is given on W an algebra structure on F by putting $ij = k = -ji$, $jk = \beta i = -kj$, $ki = -\alpha j = -ik$, $i^2 = \alpha e$, $j^2 = \beta e$ and $k^2 = -\alpha\beta e$. It can be proved (see for example [8]) that this algebra is associative, has a unit element that can be written 1, and its centre is $F \cdot 1$ which can be identified to F. This algebra is often noted as $\left(\frac{\alpha \ \beta}{F}\right)$. For example, the algebra of HAMILTON's usual quaternions is $\left(\frac{-1 \ -1}{\mathbb{R}}\right)$. It can be proved that if α and β are such that the equation $X^2 - \alpha Y^2 - \beta Z^2 + \alpha\beta T^2 = 0$ has no nontrivial solution, therefore $\left(\frac{\alpha \ \beta}{F}\right)$ is a sfield. Thus $\left(\frac{-1 \ -1}{\mathbb{R}}\right)$ is a sfield. Given $D = \left(\frac{\alpha \ \beta}{F}\right)$, any element of D is written in only one way, $a+bi+cj+dk$ where $a, b, c, d \in F$. Note σ as the transformation of D :

$$a + bi + cj + dk \mapsto a - bi - cj - dk.$$

We verify that σ is an involution of D.

b) If the characteristic of F is 2, then another associative algebra is obtained [8] by imposing the following relations on e, i, j and k : $i^2 = \alpha e$, $j^2 = j + \beta e$, $k = ij$, and $ji = k + i$. This algebra is a sfield if and only if the equation $X^2 + XY + \alpha Y^2 + \beta Y + \alpha YZ + \alpha\beta Y = 0$ does not have a nontrivial solution. The map $\sigma : a + bi + cj + dk \mapsto (a + bi + cj + dk) + c$ is an involution of D.

Example 4. The book [9] contains 593 pages of examples and theorems concerning sfields with involution.

Let us note

$$S(\sigma) = \{x \in D \mid \sigma(x) = x\}$$
$$\text{and} \quad V(\sigma) = \{x \in D \mid \sigma(x) = -x\}.$$

It is easy to calculate elements of $S(\sigma)$ and of $V(\sigma)$: $x+\sigma(x) \in S(\sigma)$ and $x - \sigma(x) \in V(\sigma)$ for any $x \in D$. We verify that $\sigma(F) = F$ and we put $F_0 = F \cap S(\sigma)$. We verify that F_0 is a subfield of F and that $S(\sigma)$ and $V(\sigma)$ are vector spaces on F_0. For any $x \in D$,

$$x = \tfrac{1}{2}\big(x + \sigma(x)\big) + \tfrac{1}{2}\big(x - \sigma(x)\big), \tag{2}$$

from which can be deduced the following equality of vector spaces on F_0 :

$$D = S(\sigma) \oplus V(\sigma). \tag{3}$$

We say that σ is a *central* involution if $S(\sigma) = F_0$.

For a central involution, (3) we therefore write

$$D = F_0 \oplus V(\sigma),$$

or, without any fear of confusion,

$$D = F_0 \oplus V. \tag{4}$$

The involution σ is said to be of *the first kind* if the restriction of σ to F is the identity and otherwise it is of *the second kind*.

Thus the involutions in Examples 1 and 3 are of the first type, the involution in Example 2 is of the second type.

If $x \in S(\sigma)$, then $x = \frac{1}{2}(x + \sigma(x))$, therefore in characteristic different from 2, the *reflexive* sfields as understood by DIEUDONNÉ are those sfields having a central involution of the first type (contrary to in characteristic 2: we verify that the sfields defined in Examples 3 b) are reflexive, but if D is such a sfield, its elements fixed by σ are the elements $a + bi + dk$).

Given a central involution σ and $v \in V : \sigma(v) = -v$. Then $\sigma(v^2) = v^2$, then

$$v^2 \in F_0 \tag{5}$$

1.3 The object of this paper therefore is to present three proofs of the Theorem of MOORE-DIEUDONNÉ:

Sfields of characteristic different from 2 *with a central involution are sfields described in Examples 1, 2 and 3.* (6)

2 Outline of Dieudonné's proof

Given D reflexive to the involution σ. Given $x \in D$. Since

$$x^2 - x(x + \sigma(x)) + x\sigma(x) = 0,$$

and $x + \sigma(x)$ and $x\sigma(x)$ belong to F, any element of D has therefore a degree at most 2. Given $x, y \in D$, we have

$$xy + yx = (x + y)^2 - x^2 - y^2,$$

hence

$$xy + yx = ax + by + c,$$

where $a, b, c \in F$, and:

$$yx = -xy + ax + by + c. \tag{7}$$

Given $x_1, \ldots, x_n \in D$ and A the sub-F-algebra of D generated by $x_1, \ldots, x_n$. Therefore A is the set of the sums of elements of the form

$$\alpha y_1^{i_1} \cdots y_m^{i_m},$$

in which $\alpha \in F$ and $y_\lambda \in \{x_1, \ldots, x_n\}$. According to (7), all the elements of A can be expressed as the sum of elements of the form

$$\alpha x_1^{i_1} \cdots x_n^{i_n},$$

in which the $i_\lambda \in \{0, 1\}$. The monomials $x_1^{i_1} \cdots x_n^{i_n}$ are finite in number. Hence A, as a F-vector space, is finite-dimensional. Since A is an algebra on F, without zero-divisor (since A is a sub-ring of a sfield) and of finite dimension on F, A is a sfield [10, p. 31] on finite rank on its center C (which contains F). Therefore [11] this rank $[A : C]$ is the square m^2 of an integer m, and A contains a separable commutative sub-field L of rank m on C.

Therefore, by the theorem of the primitive element [12], $L = C(y)$, for some a $y \in L$. DIEUDONNÉ proves [2, p. 14] that $C = F$ by using the fact that if k and K are commutative sfields such that $K = k(\alpha, \beta)$ where α is separable on K, then there is $\gamma \in K$ for which $K = k(\gamma)$ [13, p. 54]. Therefore $L = F(y)$, where y is of degree at most 2 on F. Therefore $[A : C]$ equals 1 or 4. From this, it can be deduced that $[D : F]$ is 1 or 4, and D is either F or a sfield of quaternions on F with the canonical involution (described in Example 3).

3 Moore's proof

3.1 Useful equalities. In this paragraph, taking MOORE's notations into account, α and a represent any elements of F and D respectively, and v, v_i, v', v'' any elements of V. Define $S(a)$ and $V(a)$ by: $S(a) = \frac{1}{2}(a + \sigma(a))$ and $V(a) = \frac{1}{2}(a - \sigma(a))$. Therefore the equality (2) can be written

$$a = S(a) + V(a). \tag{8}$$

Let us right multiply the two members of (8) by v :

$$av = S(a)v + V(a)v.$$

Similarly

$$va = S(a)v + vV(a).$$

Since $S(a)v \in V$, it can be deduce from (4) that:

$$S(av) = S\big(V(a)v\big) \quad \text{and} \quad S(va) = S\big(vV(a)\big). \tag{9}$$

Given $v_1, \dots, v_n \in V$. Then

$$\begin{aligned}\sigma(v_1 \cdots v_n) &= \sigma(v_n) \cdots \sigma(v_1) \\ &= (-v_n) \cdots (-v_1) \\ &= (-1)^n \, v_n \cdots v_1.\end{aligned}$$

Hence, by the definition of $S(a)$:

$$\begin{aligned}S(v_1 \cdots v_n) &= \tfrac{1}{2}\big(v_1 \cdots v_n + \sigma(v_1 \cdots v_n)\big) \\ &= \tfrac{1}{2}\big(v_1 \cdots v_n + (-1)^n v_n \cdots v_1\big),\end{aligned}$$

therefore

$$S(v_n \cdots v_1) = (-1)^n S(v_1 \cdots v_n).$$

We shall use the particular cases:

$$S(v_2 v_1) = S(v_1 v_2) \tag{10}$$

$$\text{and} \qquad S(v_3 v_2 v_1) = -S(v_1 v_2 v_3). \tag{11}$$

In the same way, it can be proved that

$$V(v_n \cdots v_1) = (-1)^{n+1} V(v_1 \cdots v_n),$$

and we will use

$$V(v_2v_1) = -V(v_1v_2) \tag{12}$$

$$\text{and} \qquad V(v_3v_2v_1) = V(v_1v_2v_3). \tag{13}$$

By the definition of $S(a)$:

$$S(vv') = \tfrac{1}{2}(vv' + v'v),$$

therefore

$$vv' = 2S(vv') - v'v. \tag{14}$$

We then obtain successively

$$\begin{aligned}
V(v_1v_2) &= \tfrac{1}{2}(v_1v_2 - v_2v_1),\\
V(v_1v_2)v_3 &= \tfrac{1}{2}(v_1v_2v_3 - v_2v_1v_3)\\
&= \tfrac{1}{2}\big(v_1(2S(v_2v_3) - v_3v_2) - v_2(2S(v_1v_3) - v_3v_2)\big) \quad \text{by (14)},\\
&= \tfrac{1}{2}\big(2S(v_2v_3)v_1 - 2S(v_1v_3)v_2 - v_1v_3v_2 + v_2v_3v_1\big),
\end{aligned}$$

therefore

$$V\big(V(v_1v_2)v_3\big) = S(v_2v_3)v_1 - S(v_1v_3)v_2 + \tfrac{1}{2}V(v_2v_3v_1 - v_1v_3v_2).$$

But (13) says the last term of the second member is zero, therefore

$$V\big(V(v_1v_2)v_3\big) = S(v_2v_3)v_1 - S(v_1v_3)v_2. \tag{15}$$

Let us replace v_3 by $V(v_3v_4)$ in this last equality:

$$V\big(V(v_1v_2)V(v_3v_4)\big) = S\big(v_2V(v_3v_4)\big)v_1 - S\big(v_1V(v_3v_4)\big)v_2. \tag{16}$$

But by (12), the first member of (16) equals $-V\big(V(v_3v_4)V(v_1v_2)\big)$, which equals, as for (16)

$$-\big(S(v_4V(v_1v_2))v_3 - S(v_3V(v_1v_2))v_4\big).$$

Replacing the first member of (16) by this last expression, we get

$$\begin{aligned} S\big(v_2V(v_3v_4)\big)v_1 - S\big(v_1V(v_3v_4)\big)v_2 \\ + S\big(v_4V(v_1v_2)\big)v_3 - S\big(v_3V(v_1v_2)\big)v_4 = 0. \end{aligned} \tag{17}$$

By (9):

$$S\big(vV(v'v'')\big) = S(vv'v''),$$

therefore (17) can be written

$$S(v_2v_3v_4)v_1 - S(v_1v_3v_4)v_2 + S(v_4v_1v_2)v_3 - S(v_3v_1v_2)v_4 = 0. \tag{18}$$

It will be more convenient to write otherwise this last equality:

$$\begin{aligned} S(v_1v_3v_4) &= -S(v_4v_3v_1) && \text{by (11)} \\ &= -S\big(v_4V(v_3v_1)\big) && \text{by (9)} \\ &= -S\big(V(v_3v_1)v_4\big) && \text{by (10)} \\ &= -S(v_3v_1v_4) && \text{by (9).} \end{aligned}$$

It can be even calculated that $S(v_4v_1v_2) = S(v_1v_2v_4)$ and $S(v_3v_1v_2) = S(v_1v_2v_3)$. Equality (18) then becomes :

Lemma 1. *If v_1, v_2, v_3 and v_4 are elements of V, then*

$$S(v_1v_2v_3)v_4 = S(v_2v_3v_4)v_1 + S(v_3v_1v_4)v_2 + S(v_1v_2v_4)v_3. \tag{19}$$

3.2 Moore's proof. Given (4). Three cases arise :

1) $V = \{0\}$. Hence $D = F_0 = F$. See Examples 1.

2) $V \ni v_1 \neq 0$. Hence

a) either $\forall v : V(v_1 v) = 0$, but then for any a:

$$V\big(v_1 V(a)\big) = 0,$$

Therefore by (4):

$$v_1 V(a) \in F_0.$$

Let us put $v_1 V(a) = s_a$, then:

$$v_1^2 V(a) = s_a v_1,$$

but $v_1^2 \in F_0$ (by (5)), so

$$V(a) = (v_1^2)^{-1} s_a v_1,$$

and

$$a = S(a) + (v_1^2)^{-1} s_a v_1,$$

therefore

$$D = F_0 + F_0 v_1.$$

However $v_1 \notin F_0$, therefore 1 and v_1, are linearly independent on F_0, so D is a commutative sfield, a quadratic extension of F_0 (Examples 2);

b) or else there is a v_0 for which $V(v_1 v_0) \neq 0$. Let us then put

$$v_2 = v_0 - S(v_1 v_0)(v_1^2)^{-1} v_1. \tag{20}$$

Let us multiply on the left the two members of this equality by v_1:

$$\begin{aligned} v_1 v_2 &= v_1 v_0 - S(v_1 v_0) \\ &= V(v_1 v_0). \end{aligned} \tag{21}$$

But, by hypothesis $V(v_1v_0) \neq 0$, therefore $v_2 \neq 0$. Let us put $v_3 = v_1v_2$. By (21), $v_3 \in V$, therefore $\sigma(v_3) = -v_3 = -v_1v_2$. But $\sigma(v_3) = \sigma(v_1v_2) = \sigma(v_2)\sigma(v_1) = v_2v_1$, therefore

$$v_2v_1 = -v_1v_2, \tag{22}$$

thus

$$\begin{aligned} v_3^2 &= v_1v_2v_1v_2 \\ &= -v_1^2v_2^2 \qquad \text{by (22)} \end{aligned}$$

Let us then put $n_1 = -v_1^2$, $n_2 = -v_2^2$, $n_3 = -v_3^2 = n_1n_2$. We verify that

$$v_3v_1 = n_1v_2 = -v_1v_3$$

and

$$v_3v_2 = -n_2v_1 = -v_2v_3.$$

We deduce from this that v_1, v_2 and v_3 are linearly independent on F_0 : given $\alpha_1, \alpha_2, \alpha_3 \in F_0$ for which

$$\alpha_1v_1 + \alpha_2v_2 + \alpha_3v_3 = 0.$$

Let us multiply this equality on the left by v_1 :

$$\alpha_1v_1^2 + \alpha_2v_1v_2 + \alpha_3v_1v_3 = 0,$$

hence

$$\alpha_1v_1^2 + \alpha_2v_3 - \alpha_3n_1v_2 = 0.$$

By (4) $\alpha_1 = 0$, therefore

$$\alpha_2v_3 - \alpha_3n_1v_2 = 0,$$

therefore

$$\alpha_2 v_3^2 - \alpha_3 n_1 v_2 v_3 = 0,$$

therefore

$$\alpha_2 v_3^2 - \alpha_3 n_1 n_2 v_1 = 0,$$

therefore

$$\alpha_2 = 0 = \alpha_3.$$

By the lemma 1, $F_0 v_1 + F_0 v_2 + F_0 v_3 = V$: indeed

$$v_1 v_2 v_3 = v_3^2 \in F,$$

and

$$S(v_1 v_2 v_3) = v_3^2 \neq 0.$$

Hence by (19), any element v_4 of V is a linear combination of v_1, v_2, v_3.

(MOORE specifies:

$$v = -\frac{S(v_1 v)}{n_1} v_1 - \frac{S(v_2 v)}{n_2} v_2 - \frac{S(v_3 v)}{n_3} v_3,$$

therefore by (9), for $a \in D$:

$$a = S(a) - \frac{S(v_1 V(a))}{n_1} v_1 - \frac{S(v_2 V(a))}{n_2} v_2 - \frac{S(v_3 V(a))}{n_3} v_3 \,.)$$

By (4), D is then the sfield of quaternions $\left(\frac{-n_1 \,,\, -n_2}{F_0}\right) = \left(\frac{-n_1 \,,\, -n_2}{F}\right)$ described in Example 3. □

3.3 **Remark 1.** MOORE thus bases his proof on the $V(vv')$, i.e. on the "oriented areas". But for his definition of v_2 (20), it is difficult not to think of GRAM-SCHMIDT's argument as applied to the scalar product $S(vv')$. The proof shown in point 4 is based on this scalar product.

Remark 2. Equality (19) has been written for the quaternions of $\left(\frac{-1 \quad -1}{\mathbb{R}}\right)$ by HAMILTON [6, p. 317]. The link between the usual quaternions and the scalar and vector products of the usual space oriented in the appropriate way is provided by the formula

$$v_1 v_2 = -\langle v_1, v_2 \rangle + v_1 \times v_2,$$

where v_1 and v_2 are pure quaternions ($\sigma(v_i) = -v_i$). Whence $S(v_1 v_2) = -\langle v_1, v_2 \rangle$ and $V(v_1 v_2) = v_1 \times v_2$. Hence the equality (15) is a translation of

$$v_3 \times (v_1 \times v_2) = \langle v_2, v_3 \rangle v_1 - \langle v_1, v_3 \rangle v_2.$$

Similarly (18) is a translation of (1).

HAMILTON points out (also on p. 317 of [6]) that (19) implies that if v_1, v_2 and v_3 are non coplanars vectors, then any vector is a linear combination of these three vectors. Furthermore, the second following equality

$$S(v_1 v_2 v_3) = S\big(v_1 V(v_2 v_3)\big) = S\big((v_1 v_2) v_3\big)$$

is translated by

$$\langle v_1, v_2 \times v_3 \rangle = \langle v_1 \times v_2, v_3 \rangle$$

and is, modulo the sign, the volume of the parallelipiped on v_1, v_2 and v_3.

The forthcoming article by J.P. MORANDI [14] links an extension of the cross product to LIE and composition algebras.

Remark 3. MOORE has thus defined [3, 4] a determinant for hermitian quaternionic matrices.

His determinant has been rediscovered at least twice and the links with other determinants, including that of DIEUDONNÉ have been studied [15, 16]. For more information, see [17] and [18]. More recently, J.P. TIGNOL has proved [19] that MOORE's determinant is a particular case of the reduced pfaffian norm introduced by M.-A. KNUS, R. PARIMALA and R. SRIDHARAN.

4 Proof based on scalar products

4.1 Let W be a *symmetric bilinear space* on F, i.e. W is a vector space on F with a symmetric bilinear map $W \times W \to F$. Let $x, y \in W$. If $b(x,y) = 0$, it is written $x \perp y$ and x and y are said to be orthogonal. If $x \neq 0$ and $b(x,x) \neq 0$, x is said to be anisotropic. The family $\{x_1, \ldots, x_n\}$ of elements of W is said to be *orthogonal* if $x_i \perp x_j$ when $i \neq j$.

It is clear that

> *if the family $\{x_1, \ldots, x_n\}$ is an orthogonal family of elements all of which are anisotropic, then $x_1, \ldots, x_n$ are linearely independent on F.* (23)

It can be proved elementarily (see e.g. [20], pp. 129–130) that any bilinear symmetric space W has a base which is an orthogonal family of W. The bilinear symmetric space W is said to be *regular* if for any $0 \neq x \in W$, there is a $y \in W$ for which $b(x,y) \neq 0$. Given a bilinear symmetric and regular space and $\{x_1, \ldots, x_n\}$ an orthogonal basis of W. Hence all the x_i are anisotropic (if for example x_1 was isotropic, it would be orthogonal to all the x_i, and by linearity to W, which would contradict the regularity). The statement can be deduced:

If W is a regular symmetric space of dimension greater than 1, then anisotropic and orthogonal $x, y \in W$ exist. (24)

4.2 Lemma 2. *If W is a vector subspace of an associative algebra on F, with a unit element 1, and such that*

(i) $uv + vu \in F.1$ for any $u, v \in W$,

(ii) if $u, v \in W$ and $uv + vu = 0$, then $uv \in W$,

(iii) if $u \in W$ and $uv + vu = 0$ for any $v \in W$, then $u = 0$,

then the dimension of W is 0, 1 or 3.

Proof. With the map $b : (u, v) \mapsto uv + vu$, W is a symmetric bilinear space for which

the relation $u \perp v$ is translated by $uv + vu = 0$. (25)

Similarly $u \neq 0$ is anisotropic if and only if $u^2 \neq 0$. Relation (iii) means that W is a regular space.

If $W = \{0\}$ or if $W = Fu$, with $0 \neq u^2 \in F$, there is nothing to prove. Given $\dim_F W > 1$. By (24), there are anisotropic and orthogonal u and $v \in W$. Given $w = uv$. By (25) and (ii): $w \in W$. Hence

$$\begin{aligned} w^2 &= (uv)(uv) = u(vu)v = -u(uv)v \\ &= -u^2v^2 \neq 0 \qquad \text{(because } u \text{ and } v \text{ are anisotropic).} \end{aligned}$$

Furthermore

$$wu = (uv)u = u(vu) = -u(uv) = -uw,$$

therefore $u \perp w$. We verify similarly that $v \perp w$. Therefore since u, v and w form an orthogonal family of anisotropic vectors, they are linearely independent (by (23)). Given $W_0 = Fu + Fv + Fw$. We shall show that $W_0 = W$. Let us suppose the contrary, and given $z \in W - W_0$.

Let us note

$$t = z - \frac{b(z,u)}{u^2}u - \frac{b(z,v)}{v^2}v - \frac{b(z,w)}{w^2}w.$$

Since $z \notin W_0$, $t \neq 0$. We verify that $t \perp u$, $t \perp v$ and $t \perp w$, therefore

$$tu = -ut \tag{26}$$
$$tv = -vt \tag{27}$$
$$tw = -wt. \tag{28}$$

But

$$\begin{aligned} tw &= t(uv) && \text{(by definition of } w) \\ &= (tu)v && \text{(by the associative hypothesis)} \qquad (29) \\ &= -(ut)v && \text{by (26)} \\ &= uvt && \text{by (27)} \\ &= wt. && \qquad (30) \end{aligned}$$

Since $w \neq 0$, (28) and (30) imply $t = 0$ and therefore a contradiction. Therefore $W = W_0$. □

Proof of (6). Given therefore (4) and let us prove that V as a vector space on F_0 satisfies the hypothesis of Lemma 2:

(i) if $v_1, v_2 \in V$, then $\sigma(v_1v_2 + v_2v_1) = v_2v_1 + v_1v_2$, therefore

$$v_1v_2 + v_2v_1 \in S(\sigma) = F_0.$$

(ii) Let $v_1, v_2 \in V$ with $v_1v_2 + v_2v_1 = 0$.

$$\begin{aligned}\sigma(v_1v_2) &= \sigma(v_2)\sigma(v_1) \\ &= v_2v_1 && \text{because } v_1, v_2 \in V \\ &= -v_1v_2 && \text{by the hypothesis.}\end{aligned}$$

Therefore

$$v_1v_2 \in V.$$

(iii) Let $0 \neq v \in V$. Therefore

$$0 \neq 2v^2 = vv + vv.$$

Lemma 2 implies then $\dim_{F_0} W \in \{0, 1, 3\}$, which correspond to the cases

- $V = \{0\}$, $D = F_0 = F$, and σ is the identity map on D;
- $V = F_0u$ where $u^2 \in F_0$, hence $D = F_0 + F_0u$ is a quadratic extension of F_0, σ is the automorphism of D described in Examples 2;
- $V = F_0u + F_0v + F_0w$, with $u^2, v^2 \in F_0$ and $vu = -uv$. Hence D is the sfield $\left(\frac{u^2 \quad v^2}{F_0}\right)$, with $F_0 = F$, and σ is the involution described in Examples 3. □

4.3 Remark 4. Lemma 2 is a tool for the proof of the result of [21] which characterises the sfields of quaternions in characteristic unequal to 2 in this way : the sfield Δ with centre F is a sfield of quaternions on F if and only if $W = \{x \in \Delta - F \mid x^2 \in F\} \cup \{0\}$ is a group of more than one element, with respect to the addition. If W satisfies that last hypothesis it is clear that it is a vector space on F which also satisfies the hypotheses of Lemma 2. We then apply the Theorem of CARTAN-BRAUER-HUA (elementary proofs of which can be found, for example, in [22, p. 186] and in [23, p. 433]): *if* K *is a sfield with centre* k *and* L *is a sub-sfield of* K*, not contained in* k *and invariant by all the inner automorphisms of* K*, then* $L = K$*.* It is obvious that W is invariant by all the inner automorphisms of Δ, $W \not\subset F$, therefore the sub-sfield generated by F and W is Δ. Lemma 2 implies $\dim_F W \in \{0,1,3\}$, and as in the proof of (6) in 4.2, the fact that Δ is a non-commutative sfield implies $\dim_F W = 3$ and $F + W$ is an algebra which is a sfield of quaternions, therefore

$$\Delta = F + W.$$

It can be shown [24] that the above hypothesis on W implies that characteristic of Δ is different from 2.

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Paul.VanPraag@umh.ac.be
Université de Mons-Hainaut
Institut de Mathématique
"Le Pentagone", Avenue du Champ de Mars, 6
B-7000 Mons (Belgique)

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