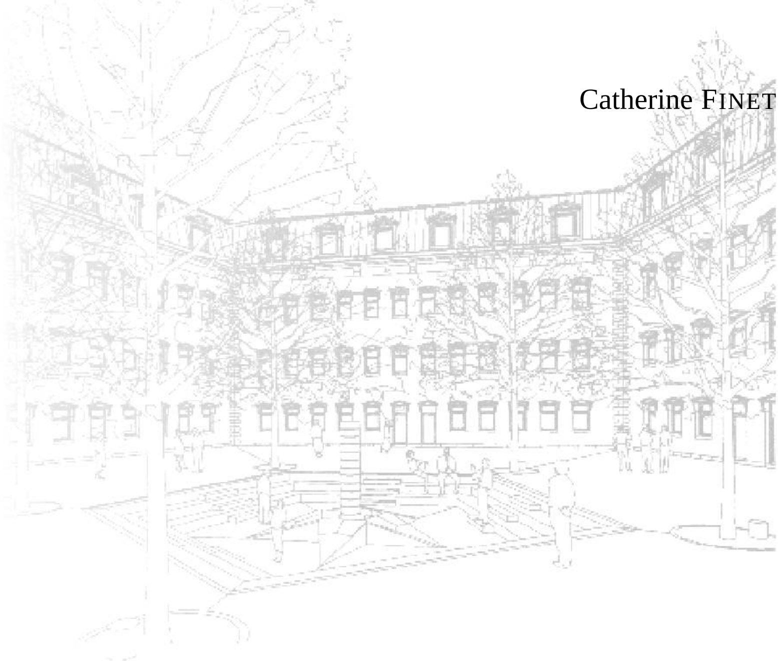
Preprint #2
June 29, 2000

Perturbed minimization principles in partially ordered Banach spaces

Catherine FINET



Institut de Mathématique et d'Informatique
Université de Mons-Hainaut
Phone: +32 65 37 35 07 — Fax: +32 65 37 33 18
Web: `http://www.umh.ac.be/math/institut`

Perturbed minimization principles in partially ordered Banach spaces

Catherine Finet

Université de Mons-Hainaut
Institut de Mathématique et d'Informatique,
« Le Pentagone », Avenue du Champ de Mars, 6
B 7000 Mons (Belgique)
e-mail: `catherine.finet@umh.ac.be`

Abstract. We give a new vector-valued extension of Deville-Godefroy-Zizler perturbed minimization principle. The functions we are considering are taking values in a real Banach space partially ordered by a closed convex pointed cone.

Mathematics Subject Classification. 46B20, 49N.
Key words and phrases. Lower semi-continuous function, Vector-valued perturbed minimization principles.

1 Introduction

Variational principles (or perturbed minimization principles) are efficient tools to deal with minimization problems without compactness. They have many applications for example in Banach space theory, potential theory, non smooth analysis, and non linear analysis. These principles consist in perturbing a bounded below, lower semi-continuous function f on an infinite dimensional Banach space X by a “good” function g in such a way that $f + g$ attains a (strong) minimum on X (every minimizing sequence converges). These principles have been studied for real-valued functions, for example in [2, 4, 6, 7, 9] and for vector-valued functions, for example in [1, 5, 10, 12, 16, 17]. In [5], we were concerned with a vector-valued extension of Deville-Godefroy-Zizler perturbed minimization principle [7], the function f taking values in a real Banach space Z partially ordered by a closed convex pointed cone with *non empty interior*.

But in many cases, the interior of the ordering cone is empty. For example, take $Z = L^p(\Omega)$, where Ω is a non empty subset of $\mathbb{R}^n$ and $p \in [1, +\infty)$, then the natural ordering cone has an empty interior.

We extend our previous approach to Banach spaces Z partially ordered by a closed convex pointed cone K. We also get a porosity result.

This answers a question asked by R. Paya during the workshop in the geometry of Banach spaces, hold at the university of Murcia (July 1999).

In the same context we also get a vector-valued extension of Ekeland variational principle and of Stegall variational principle.

2 Preliminaries

Let X and Z be real Banach spaces and Z be partially ordered by a closed convex pointed cone K. Let f be a function from X to Z. The vector minimization problem (P) under consideration is:

$$\text{Find } \overline{x} \in X \text{ such that } \big(f(\overline{x}) - K \setminus \{0\}\big) \cap f(X) = \varnothing.$$

Such an $\overline{x}$ is called an efficient solution of (P).

We denote by $E(f)$ the set of all efficient solutions, and by $\operatorname{Min} f$ the set $f(E(f))$.

Let us recall that a function $f : X \to Z$ is said *bounded below* if there exists $z \in Z$ such that for all $x \in X$, $z \leqslant f(x)$.

In [16] (see also [3]) the authors introduced the two following notions: a function $f : X \to \overline{Z} := Z \cup \{+\infty\} \cup \{-\infty\}$ is said *lower semi-continuous* (l.s.c.) at $x_0 \in X$, if for each neighborhood V of $f(x_0)$ in $\overline{Z}$ there exists a neighborhood U of x_0 in X such that $f(U) \subset V + \big(K \cup \{+\infty\}\big)$.

A function $f : X \to \overline{Z}$ is said *quasi lower semi-continuous* (quasi-l.s.c.) at $x_0 \in X$, if for each $b \in Z$ such that $b \not\geqslant f(x_0)$ there exists a neighborhood U of x_0 in X such that $b \not\geqslant f(x)$ for each x in U.

A function f is l.s.c. (resp. quasi-l.s.c.) if f is l.s.c. (resp. quasi-l.s.c.) at each point of X.

Let us recall a few facts concerning these notions [16, 3].

A l.s.c. function at x_0 is quasi-l.s.c. at x_0.

A function f is quasi-l.s.c. if and only if for each $b \in Z$ the set $\{x \in X;\ f(x) \leqslant b\}$, also denoted by $\{f \leqslant b\}$, is closed in X.

Let us remark that the notions of l.s.c. and quasi-l.s.c. coincide for real-valued functions, but it is not the case in general: take $Z = \mathbb{R}^2$, $K = \mathbb{R}^2_+$. Then the function $f : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\begin{cases} f(x) = (-1, 1/|x|), & \text{for } x \neq 0 \\ f(0) = (0, 0). \end{cases}$$

is quasi-l.s.c. but not l.s.c. at 0.

Moreover, any l.s.c. function $f : X \to \overline{Z}$ has a closed epigraph $\mathrm{Epi}(f) = \{(x, y) \in X \times Z;\ y \geqslant f(x)\}$. But the converse is not true in general as shown by the previous function f.

If $\mathrm{Epi}(f)$ is closed then f is quasi-l.s.c.; the converse is true if the interior of K is non empty. Let us also recall the following properties.

LEMMA 2.1. ([16]) *The sum of two l.s.c. functions is l.s.c.*

Let G be an ordered topological vector space (o.t.v.s.) then a function $g : Z \to G$ is said *monotone* if $z_1, z_2 \in Z$ and $z_1 \geqslant z_2$ then $g(z_1) \geqslant g(z_2)$.

LEMMA 2.2. ([16]) *Let* $f : X \to Z$ *be l.s.c. and let* $g : Z \to G$ *be a monotone l.s.c. function with values in an o.t.v.s.* G*. Then* $g \circ f$ *is l.s.c.*

In [5] we established a vector-valued Deville-Godefroy-Zizler perturbed minimization principle for bounded below, quasi-l.s.c. functions with values in a real Banach space Z partially ordered by a closed convex pointed cone with *non empty interior*. Here we get a vector-valued Deville-Godefroy-Zizler perturbed minimization principle for bounded below, l.s.c. functions with values in a real Banach space Z partially ordered by a closed convex pointed cone K.

Of course now our principle applies to $Z = L^p(\Omega)$, $p \in [1, \infty)$, Ω a non empty subset of $\mathbb{R}^n$.

3 Vector-valued Deville-Godefroy-Zizler perturbed minimization principle

Let e be in $K \setminus \{0\}$ then as K is pointed it follows that $-e \notin K$. Then by Hahn-Banach theorem there exists $e^\star \in K^\star$, the dual cone of K $(K^\star = \{z^\star \in Z^\star;\ z^\star(z) \geqslant 0,\ \forall z \in K\})$ such that $e^\star(e) > 0$. In what follows $e^\star$ (resp. e) will be a fixed element in $K^\star$ (resp. K) such that $e^\star(e) = 1$. Let us point out some properties of $e^\star$. Let $f : X \to Z \cup \{+\infty\}$. We put $e^\star(+\infty) = +\infty$.

(1) If f is bounded below, it follows, from the monotonicity of $e^\star$ that $e^\star \circ f : X \to \mathbb{R} \cup \{+\infty\}$ is also bounded below.

(2) If f is l.s.c. then $e^\star \circ f$ is l.s.c. (see Lemma 2.2).

(3) Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a function and $x_0 \in X$ be such that $g(x_0) < \infty$, then we say that g has a *strong minimum* on X at x_0 if $g(x_0) = \inf\{g(x);\ x \in X\}$ and if $(x_n) \subset X$ is such that $g(x_n) \to g(x_0)$ then $\|x_n - x_0\| \to 0$ (that is, every minimizing sequence converges to x_0). Note that, in particular, x_0 is then a unique minimum. And by a remark of L. Quarta and C. Troestler if $e^\star \circ f$ attains its strong minimum at $x_0 \in X$ then by the monotonicity of $e^\star$, we have that $x_0 \in E(f)$.

We call a *bump function* on X a real-valued function on X with bounded non empty support. Let us denote for $f : X \to Z$, $\|f\|_\infty = \sup\limits_{x \in X} \|f(x)\|$.

THEOREM 3.3. *Let $(Y, \|\cdot\|_Y)$ be a complete convex cone of norm bounded, bounded below, l.s.c. functions from X to Z such that*

- *(i) for all $g \in Y$, $\|g\|_\infty \leqslant \|g\|_Y$;*
- *(ii) Y is translation invariant, i.e., if $g \in Y$ and $x \in X$, then $\mathcal{T}_x g : X \to Z$ given by $\mathcal{T}_x g(t) = g(x+t)$ is in Y and $\|\mathcal{T}_x g\|_Y = \|g\|_Y$;*
- *(iii) Y is dilatation invariant, i.e., if $g \in Y$ and $\alpha \in \mathbb{R}$ then $g^\alpha : X \to Z$ given by $g^\alpha(t) = g(\alpha t)$ is in Y;*
- *(iv) There exists a bump function $b : X \to \mathbb{R}$ such that $b(0) > 0$ and $\tilde{b} = -be$ belongs to Y.*

Let $f : X \to Z \cup \{+\infty\}$ be a l.s.c., bounded below function such that $D(f) = \{x \in X;\ f(x) \in Z\} \neq \varnothing$. Then the set of all $g \in Y$ such that $e^\star \circ (f + g)$ attains its strong minimum on X is a dense G_δ subset G of Y.

PROOF. The proof follows the ideas used by Deville-Godefroy-Zizler [7] and Deville-Finet [5]. We consider the set $U_n := \big\{g \in Y;\ \exists x_n \in X,\ e^\star(f+g)(x_n) < \inf\{e^\star(f+g)(x),\ x \notin B(x_n, 1/n)\}\big\}$, where $e^\star(f+g)(x) = e^\star((f+g)(x))$. Since f and g are bounded below, by monotonicity of $e^\star$, we have $e^\star \circ (f+g)$ is bounded below. We now prove that U_n is an open set. Let g be in U_n, then there exists $x_n \in X$ such that $\varepsilon = \inf\big\{e^\star(f+g)(x),\ x \notin B(x_n,1/n)\big\} - e^\star(f+g)(x_n) > 0$. Let $h \in Y$. Then

$$\sup_{x \in X} |e^\star(h)(x)| \leqslant \|e^\star\|\, \|h\|_\infty \leqslant \|e^\star\|\, \|h\|_Y \qquad (\star)$$

Let $k \in Y$ be such that $\|g - k\|_Y \leqslant \varepsilon/4\|e^\star\|$. We have $e^\star(f+k)(x_n) = e^\star(f+g)(x_n) + e^\star(k-g)(x_n)$.

By definition of ε, for $x \notin B(x_n, 1/n)$ and by $(\star)$

$$\begin{aligned} e^\star(f+k)(x_n) &\leqslant e^\star(f+g)(x) - 3\varepsilon/4 \\ &\leqslant e^\star(f+k)(x) + e^\star(g-k)(x) - 3\varepsilon/4 \\ &\leqslant e^\star(f+k)(x) - \varepsilon/2 \qquad \text{(by } (\star)\text{)} \end{aligned}$$

Thus, for $x \notin B(x_n,1/n)$, $e^\star(f+k)(x_n) \leqslant e^\star(f+k)(x) - \varepsilon/2$.

This proves that $k \in U_n$.

To get that U_n is dense in Y follow the proof of [5] using the facts that $e^\star(e) = 1$ and $e^\star$ is linear. Consequently $G = \bigcap_n U_n$

is a dense G_δ subset of the complete cone Y. If $g \in G$ then $e^\star \circ (f+g)$ attains its strong minimum on X. Indeed, following Lemma 2.1, as f and g are l.s.c. then $f+g$ is l.s.c. and by property (2) of $e^\star$ it follows that $e^\star \circ (f+g)$ is l.s.c. Then following [5], we get that $e^\star \circ (f+g)$ attains its strong minimum. And conversely, if $e^\star \circ (f+g)$ attains its strong minimum then $g \in G$. ■

Let us mention that when $\operatorname{int} K \neq \varnothing$, if f is norm bounded then f is bounded below. Indeed, let $k^0 \in \operatorname{int} K$. There exists $r > 0$ such that the ball $B(k^0, r)$ is contained in K. Then for each $x \in X$, $k^0 \pm rf(x)/\|f\|_\infty \in K$ and as K is a cone,

$$f(x) \in K - k^0 \|f\|_\infty / r.$$

We now give a porosity result. Let us recall the following:

DEFINITION 3.4. ([8], SEE ALSO [19]) Let (X,d) be a metric space and A be a subset of X. The set $A \subset X$ is said to be *porous* in X if there exist $\lambda_0 \in (0,1]$ and $r_0 > 0$ such that for any $x \in X$ and $r \in (0, r_0]$ there exists $y \in X$ such that $B(y, \lambda_0 r) \subset B(x,r) \cap (X \setminus A)$.

Here $B(z,t)$ stands for the open ball in X centered at z with radius $t > 0$. The set $A \subset X$ is called *σ-porous* in X if it can be represented as a countable union of porous sets in X.

As a Lipschitz continuous function $g : X \to Z$ is l.s.c., we can take for Y the Banach space of all bounded below, norm bounded Lipschitz continuous functions from X to Z equipped with the norm:

$$\|g\|_Y = \|g\|_\infty + \sup\left\{\frac{\|g(x)-g(y)\|}{\|x-y\|};\ x \neq y\right\}.$$

Following the proof of [8] it is then easy to get

PROPOSITION 3.5. *Let X and Z be as before. Let $f : X \to Z \cup \{+\infty\}$ be a proper, norm bounded, bounded below, l.s.c. function. Then the set*

$$G := \{g \in Y;\ e^\star \circ (f+g) \textit{ attains its strong minimum on } X\}$$

is a G_δ-subset of Y whose complement in Y is σ-porous in Y. In particular the set $\{g \in Y;\ \mathrm{Min}(f+g) \neq \varnothing\}$ has a complement in Y which is σ-porous.

We now give some corollaries of Theorem 3.3. Let Y be as before then

COROLLARY 3.6. *Let $f : X \to Z$ be l.s.c., bounded below. Then, for every $\varepsilon > 0$, there exists $g : X \to Z$ bounded below, norm bounded Lipschitz continuous function such that*

$$\|g\|_Y < \varepsilon, \quad \mathrm{Min}(f+g) \neq \varnothing.$$

Take now for Y the Banach space of all $g : X \to Z$ that are bounded below, norm bounded, Lipschitz continuous and Fréchet-differentiable (resp. Gateaux-differentiable) equipped with the norm : $\|g\|_Y := \max(\|g\|_\infty, \|g'\|_\infty)$. We then get a vector-valued version of the Borwein-Preiss smooth perturbed minimization principle.

COROLLARY 3.7. (SMOOTH PERTURBED MINIMIZATION PRINCIPLE) *Let X be a Banach space that admits a Lipschitz continuous bump function which is Fréchet-differentiable (resp. Gateaux-differentiable). Then for every $f : X \to Z$ l.s.c., bounded below, and for every $\varepsilon > 0$, there exists a function $g : X \to Z$ which is Lipschitz continuous and Fréchet-differentiable (resp. Gateaux-differentiable) and such that $\|g\|_\infty < \varepsilon$, $\|g'\|_\infty < \varepsilon$, $\mathrm{Min}(f + g) \neq \varnothing$.*

4 Vector-valued Ekeland variational principle

As before let X and Z be real Banach spaces and Z be partially ordered by a closed convex pointed cone K. Let $e^\star$ (resp. e) be a fixed element in $K^\star$ (resp. K) such that $e^\star(e) = 1$. We put $e^\star(+\infty) = +\infty$.

THEOREM 4.8. *Let $f : X \to Z \cup \{+\infty\}$ be a quasi-l.s.c., bounded below function such that $D(f) \neq \varnothing$. Let $\varepsilon > 0$ and $\lambda > 0$ be given and let $x \in X$ be such that*

$$(e^\star \circ f)(x) \leqslant \inf_X (e^\star \circ f) + \varepsilon \tag{$\star$}$$

Then there exists $x_\varepsilon \in X$ such that:

(i) $f(x_\varepsilon) \leqslant f(x)$

(ii) $\|x - x_\varepsilon\| \leqslant \lambda$

(iii) $x_\varepsilon \in E(f_\lambda)$*, where* $f_\lambda(x) := f(x) + (\varepsilon/\lambda)\|x - x_\varepsilon\|e$*.*

PROOF. Define an order $<$ on X by:

$$x < y \;\Leftrightarrow\; f(x) + \frac{\varepsilon}{\lambda}\|x - y\|e \leqslant f(y).$$

Let us define inductively a sequence $(u_n)_{n\in\mathbb{N}}$ in X, starting with $u_0 = x$. Suppose $u_n \in X$ is known. Then choose $u_{n+1} \in S_n = \{w \in X,\ w < u_n\}$, such that $(e^\star \circ f)(u_{n+1}) \leqslant \inf_{S_n}(e^\star \circ f) + 1/2^n$.

As f is bounded below, there exists $z \in Z$, such that, $z \leqslant f(u_{n+1})$, and as $u_{n+1} \in S_n$, $f(u_{n+1}) \leqslant f(u_n)$. From the monotonicity of $e^\star$ it follows that the sequence $\big((e^\star \circ f)(u_n)\big)_{n\in\mathbb{N}}$ is decreasing and bounded below, hence convergent.

Let $w \in S_n$ then

$$\frac{\varepsilon}{\lambda}\|w - u_n\|e \leqslant f(u_n) - f(w),$$

by monotonicity of $e^\star$ and the fact that $e^\star(e) = 1$, we get,

$$\frac{\varepsilon}{\lambda}\|w - u_n\| \leqslant (e^\star \circ f)(u_n) - (e^\star \circ f)(w),$$

and by the choice of the sequence (u_n)

$$\frac{\varepsilon}{\lambda}\|w - u_n\| \leqslant (e^\star \circ f)(u_n) - (e^\star \circ f)(u_{n+1}) + \frac{1}{2^n}.$$

Thus, $\operatorname{diam} S_n \to 0$.

Moreover, as f is quasi-l.s.c., it follows that for each $y \in X$ the function : $x \mapsto f(x) + (\varepsilon/\lambda)\|x - y\|e$ is also quasi-l.s.c. And the set S_n is closed. As $S_{n+1} \subset S_n$, it follows that there exists $x_\varepsilon \in X$ such that $\bigcap_{n\in\mathbb{N}} S_n = \{x_\varepsilon\}$. Therefore, in particular, $v_\varepsilon \in S_0$ and $f(x_\varepsilon) \leqslant f(x)$. Moreover, the inequality $f(x_\varepsilon) \leqslant f(x) - (\varepsilon/\lambda)\|x - x_\varepsilon\|e$ implies

$$\begin{aligned}\frac{\varepsilon}{\lambda}\|x - x_\varepsilon\| &\leqslant (e^\star \circ f)(x) - (e^\star \circ f)(x_\varepsilon) \\ &\leqslant \inf_X (e^\star \circ f) + \varepsilon - (e^\star \circ f)(x_\varepsilon) \leqslant \varepsilon.\end{aligned}$$

And $\|x - x_\varepsilon\| \leqslant 1/\lambda$.

Now take $w < x_\varepsilon$, then $\forall n$, $w < u_n$ and $w \in \bigcap_n S_n$, thus $w = x_\varepsilon$, this is assertion (iii). ■

Let us mention that Ch. Tammer [17, 18] got this result, when $\operatorname{int} K \neq \varnothing$, and instead of considering points x satisfying ($\star$) she considered ε-approximatively efficient points, that are points x satisfying

$$f(X) \cap \big(f(x) - \varepsilon e - K \setminus \{0\}\big) = \varnothing. \qquad (\star\star)$$

Let us mention that ($\star$) implies ($\star\star$).

Let us also mention that if we apply the scalar-valued Ekeland variational principle to the function $e^\star \circ f$ we do not get the result. Indeed for (i) we then get $(e^\star \circ f)(x_\varepsilon) \leqslant (e^\star \circ f)(x)$, but this does not imply that $f(x_\varepsilon) \leqslant f(x)$ and for (iii) there is no reason why x_ε should be in $E(f_\lambda)$.

5 Vector-valued Stegall variational principle

THEOREM 5.9. *Let C be a non empty closed bounded subset of X with Radon Nikodym property. Let $f : C \to Z$ be a l.s.c. bounded below function. Then for every $\varepsilon > 0$, there exists $L : X \to Z$ linear continuous operator with $\|L\| \leqslant \varepsilon$ and $e^\star \circ (f + L)$ attains its strong minimum on C. In particular* $\mathrm{Min}(f + L) \neq \varnothing$.

PROOF. It suffices to show that for every $x^\star$ in $X^\star$ there exists $L : X \to Z$ linear continuous operator such that $x^\star = e^\star \circ L$. Indeed as $e^\star \circ f$ is l.s.c. bounded below we can apply Stegall variational principle [13] to the real-valued function $e^\star \circ f$. Let $\varepsilon > 0$ and $\delta \leqslant \varepsilon/\|e\|$, there exists $x^\star \in X^\star$, $\|x^\star\| \leqslant \delta$ and $e^\star \circ f + x^\star$ attains its strong minimum on C.

Now consider $L = ex^\star$, then L is a linear continuous operator, with $\|L\| \leqslant \varepsilon$ and $x^\star = e^\star \circ L$. And we get the result. ■

Acknowledgements

We want to thank P. Georgiev for the remarks he made on this paper. The author thanks “La Banque Nationale de Belgique” for the grant she got.

References

[1] M. Bachir, Dualité, calcul sous-différentiel et intégration en analyse non lisse et non convexe. Thèse de doctorat, Université de Bordeaux I, 2000.

[2] J. Borwein and D. Preiss, Smooth variational principle with applications to subdifferentiability of convex functions, *Trans. Amer. Math. Soc.* **303**, 1987, 517–527.

[3] J. Borwein, J. Penot and M. Théra, Conjugate convex operators, *J. Math. Anal. Appl.* **102**, 1984, 399–414.

[4] R. Deville. Smooth variational principles and non-smooth analysis in Banach spaces, *Nonlinear Analysis, Differential Equations and Control*, 1999, 369–405.

[5] R. Deville and C. Finet, Vector-valued perturbed minimization principles, *Canadian Math. Soc.*, Conference Proceedings (to appear).

[6] R. Deville, N. Ghoussoub. Perturbed minimization principles and applications (to appear).

[7] R. Deville, G. Godefroy and V. Zizler, Smoothness and renormings in Banach spaces, *Pitman monographs and Surveys in Pure and Appl. Math.* Longman Scientific & Technical, 1993.

[8] R. Deville and J.P. Revalski, *Porosity of ill-posed problems* (to appear).

[9] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* **47**, 1974, 324–353.

[10] A. Göpfert and Chr. Tammer, A new maximal point theorem, *Zeitschr. f. Analysis und Anwendungen*, 1994.

[11] J. Jahn, *Mathematical vector optimization in partially ordered spaces*, Lang Verlag Frankfurt, Bern, New York, 1986.

[12] P. Loridan, Well-posedness in vector optimization, *Cermsem 95. 25*.

[13] C. Stegall, Optimization of functions on certain subsets of Banach spaces, *Math. Ann.* **236**, 1978, 171–176.

[14] J. Penot and M. Théra, Applications sous-linéaires à valeurs dans un espace de fonctions continues, *Annali di Matematica Pura ed Applicata* **136**, 1984, 133–151.

[15] J. Penot and M. Théra, Semi-continuous mappings in general topology, *Archiv. der Math.* **38**, 1982, 158–166.

[16] M. Théra, Etudes des fonctions convexes vectorielles semi-continues. Thèse de 3ème cycle, Université de Pau, 1978.

[17] Chr. Tammer, A generalization of Ekeland's variational principle, *Optimization* **25**, 1992, 129–141.

[18] Chr. Tammer, A variational principle and applications for vectorial control approximation problems, Reports of the

Inst. of Optimization and Stochastics. Martin-Luther Universität Halle — Wittenberg, 1996.

[19] L. Zajíček, Porosity and σ-porosity, *Real Anal. Exchange* **13**, 1987–88, 314–350.

Recent preprints

[1] *Maurice Boffa's 60th Birthday Workshop*, March 23, 2000.

You can find more informations as well as download the preprints of the *Institut de Mathématique et d'Informatique* on the web site: `http://www.umh.ac.be/math/preprints/`. Printed copies are available upon request by writing to:

Institut de Mathématique et d'Informatique
Université de Mons-Hainaut
« Le Pentagone », 6 av. du champ de Mars
7000 Mons, Belgique