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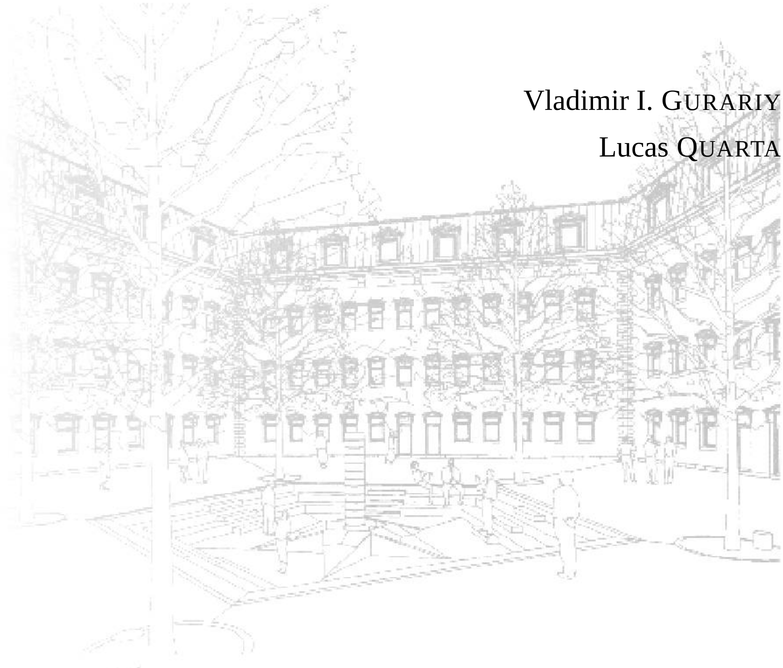

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On lineability of sets of continuous functions

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Abstract. We study the existence of vector spaces of dimension at least two of continuous functions on (subsets of) $\mathbb{R}$, every non-zero element of which admits one and only one absolute maximum.

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Introduction

In [1], the authors begin by the following: “In many different settings one encounters a problem which, at first glance, appears to have no solution at all. And, in fact, it frequently happens that there is a large linear subspace of solutions to the problem.”

A set M in a linear topological space X is said to be *n-lineable* (resp. *lineable*, resp. *spaceable*) in X if $M \cup \{0\}$ contains a vector space Y with $\dim Y = n$ (resp. $\dim Y = \dim \mathbb{N}$, resp. $\dim Y = \dim \mathbb{N}$ and Y is closed). If the maximum cardinality of such a vector space exists it is called the *lineability* of M and denoted by $\lambda(M)$. The set M is said to be *totally non lineable* or *very non linear* if $\lambda(M) \leqslant 1$. In [1], they give number of such results of “linearity in non linear problems” in many different fields of analysis (e.g. [12] and [3] concerning zeros of polynomials, [8] and [4] concerning hypercyclic operators, [1] concerning non extendible holomorphic functions,...). One of the first results in this spirit is the lineability of the set of nowhere differentiable functions on $[0,1]$, proved by the first author in [9]. This work has been intensively continued ([10, 7] which prove the spaceability, [13] which proves that any separable Banach space is isometrically isomorphic to such a subspace, [11]). Recently, several papers were devoted to the study of the lineability of sets of functions on $[0,1]$ or $\mathbb{R}$ which satisfy other special properties. For examples, P. Enflo and the first author have proved in [6] that for any infinite dimensional subspace X of the space $C[0,1]$ of continuous functions on $[0,1]$, the set of functions in X having infinitely many zeros in $[0,1]$ is spaceable in X and R. Aron, J. Seoane and the first author have shown in [2] that the set of everywhere surjective functions from $\mathbb{R}$ to $\mathbb{R}$ is lineable (in fact the lineability of this set is equal to the dimension of $\mathbb{R}$).

This article takes its place in that program. We study the following question: is it possible to find a vector space of dimension at least two of real-valued continuous functions with (except for the zero function) one and only one absolute maximum? The main results are the following:

Theorem 6. The set $\hat{C}[0,1]$ of real-valued continuous functions which admit one and only one absolute maximum is very non linear in $C[0,1]$. In other words, $\lambda(\hat{C}[0,1]) = 1$.

Theorem 9. The set $\hat{C}(\mathbb{R})$ is 2-lineable in $C(\mathbb{R})$.

Theorem 19. $\lambda(\hat{C}_0(\mathbb{R})) = 2$, where $C_0(\mathbb{R})$ is the space of continuous functions on $\mathbb{R}$ vanishing at infinity.

We have some other relative results, as the spaceability of the set of continuous and bounded functions on $\mathbb{R}$ without any absolute maximum and answers to the corresponding questions for sets of sequences. Also, we can complete some results obtained in [14] concerning the lineability of the set of continuous functions which attain their supremum norm at a unique point.

We will use the following notations for a function x belonging to $C(K)$ where K is a subset of $\mathbb{R}$: $M(x) := \sup_{t\in K} x(t)$, $m(x) := \inf_{t\in K} x(t)$, $\|x\| := M(|x|)$, $M_x := \{t \in K : x(t) = M(x)\}$, $m_x := \{t \in K : x(t) = m(x)\}$. We will denote by $\langle x, y\rangle$ the vector space generated by x and y, and by $|S|$ the cardinality of a set S.

1 The very non linearity of $\hat{C}[0,1]$.

The main tool in the proof of Theorem 6 will be the notions of *ignorability* and *fence*. Let us introduce these definitions.

Definition 1. Let $(x_i)_{i=1}^n$ be a finite set of functions in $C[0,1]$. A point t in $[0,1]$ is said to be *ignorable* for $(x_i)_{i=1}^n$ if for every set $(\alpha_i)_{i=1}^n$ of strictly positive real numbers, $t \notin M_{\sum_{i=1}^n \alpha_i x_i}$. A point t in $[0,1]$ is said to be a *fence* between t_1 and t_2 in $[0,1]$ for $(x_i)_{i=1}^n$ if $t \in]t_1,t_2[$ and t is ignorable for $(x_i)_{i=1}^n$.

Definition 2. A pair of functions $\{x,y\}$ in $C[0,1]$ is said to be *canonical* if $\exists t_x \in M_x$, $\exists t_y \in M_y$, $\exists \tilde{t} \in]t_x,t_y[: m_x = \{\tilde{t}\}$ or $m_y = \{\tilde{t}\}$.

Obviously, we have

Lemma 3. *In the canonical situation of Definition 2, $\tilde{t}$ is a fence for $\{x,y\}$ between t_x and t_y.*

Proof. Let us suppose that $m_x = \{\tilde{t}\}$. Then, $x(\tilde{t}) < x(t_y)$, $y(\tilde{t}) \leqslant y(t_y)$ and $\tilde{t} \notin M_{\alpha x+\beta y}$ for every strictly positive real numbers α and β. □

A canonical pair of functions cannot be the basis of a two-dimensional vector space V such that $V \setminus \{0\}$ is contained in $\hat{C}[0,1]$. Indeed,

Proposition 4. *For any canonical pair of functions $\{x,y\}$ in $C[0,1]$ there exist two positive real numbers α and β such that the function $\alpha x+\beta y$ has at least two absolute maxima.*

In order to prove this proposition, we will need the following

Lemma 5. *If Φ is a continuous map from $[0,1]$ to $C[0,1]$ such that for every α in $[0,1]$, M_{Φ_α} is a singleton $\{t_\alpha\}$, then the map μ defined from $[0,1]$ to $[0,1]$ by $\mu(\alpha) = t_\alpha$ is continuous.*

Proof of Lemma 5. Let us suppose that $\alpha \to \alpha_0$ and, by contradiction, let us suppose that (t_α) does not converge to t_{α_0}. Since $[0,1]$ is compact, up to a subsequence, (t_α) converges to a point $\tilde{t} \in [0,1]$. For every $\alpha \in [0,1]$, we have: $|\Phi_\alpha(t_\alpha) - \Phi_\alpha(\tilde{t})| \leqslant |\Phi_\alpha(t_\alpha) - \Phi_{\alpha_0}(t_\alpha)| + |\Phi_{\alpha_0}(t_\alpha) - \Phi_\alpha(\tilde{t})| \leqslant \|\Phi_\alpha - \Phi_{\alpha_0}\| + |\Phi_{\alpha_0}(t_\alpha) - \Phi_\alpha(\tilde{t})|$. Since Φ and Φ_{α_0} are continuous, we have: $|\Phi_\alpha(t_\alpha) - \Phi_\alpha(\tilde{t})| \to 0$ when $\alpha \to \alpha_0$. But we have also $M(\Phi_\alpha) = \Phi_\alpha(t_\alpha) \to M(\Phi_{\alpha_0}) = \Phi_{\alpha_0}(t_{\alpha_0})$ and $\Phi_\alpha(\tilde{t}) \to \Phi_{\alpha_0}(\tilde{t})$. Then, $\Phi_{\alpha_0}(\tilde{t}) = \Phi_{\alpha_0}(t_{\alpha_0}) = M(\Phi_{\alpha_0})$ and since $M(\Phi_{\alpha_0}) = \{t_{\alpha_0}\}$, we have $\tilde{t} = t_{\alpha_0}$. This concludes the proof. □

Proof of Proposition 4. Let us suppose that there exists a canonical pair of functions $\{x,y\}$ such that for every $(\alpha,\beta) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$, $M_{\alpha x+\beta y}$ is a singleton. Let us consider the map Φ defined from $[0,1]$ to $C[0,1]$ by $\Phi_\alpha = (1-\alpha)x + \alpha y$ and the map μ defined from $[0,1]$ to $[0,1]$ by $\mu(\alpha) = t_\alpha$ where $\{t_\alpha\} = M_{(1-\alpha)x+\alpha y}$. By Lemma 5, μ is continuous and by the intermediate value property (Weierstrass theorem) μ takes all the values between $\mu(0) = t_0$ and $\mu(1) = t_1$ where $\{t_0\} = M_x$ and $\{t_1\} = M_y$. This is in contradiction with Lemma 3 which asserts that there exists a fence between t_0 and t_1. This concludes the proof of Proposition 4. □

We can now prove the very non linearity of $\hat{C}[0,1]$.

Theorem 6. $\lambda(\hat{C}[0,1]) = 1$.

Proof. We want to prove that for any pair of linearly independent functions $\{x,y\}$ in $C[0,1]$ there exists (α,β) in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that the function $\alpha x+\beta y$ admits at least two absolute maxima. let us suppose that it is not true and consider x and y in $C[0,1]$ such that for every (α,β) in $\mathbb{R}^2 \setminus \{(0,0)\}$, $M_{\alpha x+\beta y}$ is a singleton. Let us define $\varepsilon(x,y) := M_x \cup M_y \cup m_x \cup m_y$. Obviously, $\varepsilon(x,y)$ contains at most four points: $|\varepsilon(x,y)| \leqslant 4$. We have to consider two cases:

- If $|\varepsilon(x,y)| \geqslant 3$, one of the four pairs of functions $\{x,y\}$, $\{x,-y\}$, $\{-x,y\}$ or $\{-x,-y\}$ is canonical and, by Proposition 4, we have a contradiction.
- If $|\varepsilon(x,y)| = 2$, using the idea of the proof of Proposition 4, we can find $\alpha \in]0,1[$ such that the pair of functions $\{x,(1-\alpha)x+\alpha y\}$ satisfies $\varepsilon(x,(1-\alpha)x+\alpha y) \geqslant 3$. And then, the first case gives the contradiction. □

Remark 7. Let us note that we can deduce from [14] that $\lambda(\hat{C}[0,1]) \leqslant 2$ and, actually, even more: the subset $\|\hat{C}[0,1]\|$ of $C[0,1]$ of functions which attain their supremum norm at a unique point is very non linear. This approach is connected with the existence of alternating elements in subspaces of $C[0,1]$.

2 The lineability of $\hat{C}(\mathbb{R})$.

We will prove that the situation of a close interval of the previous section is rather different from the situation of open or semi-open intervals.

Proposition 8. *$\hat{C}([0,2\pi[)$ is 2-lineable.*

Proof. Let us consider the trigonometric functions sine and cosine defined on the semi-open interval $[0,2\pi[$. We have: $\forall(\alpha,\beta) \in \mathbb{R}^2 \setminus \{(0,0)\}, \ \exists\theta \in [0,\pi], \ \alpha\cos+\beta\sin = \sqrt{\alpha^2+\beta^2}\cos(\cdot+\theta)$. Since the function cosine admits one and only one maxima on $[0,2\pi[$, this proves that $\langle\sin,\cos\rangle \setminus \{0\} \subset \hat{C}([0,2\pi[)$ and concludes the proof. □

We can now easily prove the

Theorem 9. *$\hat{C}(\mathbb{R})$ is 2-lineable.*

Proof. The functions x and y defined on $\mathbb{R}$ by

$$x(t) := \mu(t)\cos\big(4\arctan|t|\big) \quad \text{and}$$
$$y(t) := \mu(t)\sin\big(4\arctan|t|\big)$$

where μ is the real-valued continuous function defined on $\mathbb{R}$ by

$$\mu(t) := \begin{cases} \exp t & \text{if } t \leqslant 0 \\ 1 & \text{if } t \geqslant 0, \end{cases}$$

are two linearly independent functions of $C(\mathbb{R})$ such that for every (α,β) in $\mathbb{R}^2 \setminus \{(0,0)\}$: $M_{\alpha x+\beta y}$ is a singleton. □

Let us remark that the two-dimensional subspace construct in this proof is isometric to $\ell_2(2)$. It is impossible to find such a subspace isometric to $\ell_1(2)$.

Proposition 10. *It is impossible to find a two-dimensional subspace E of $C(\mathbb{R})$ isometric to $\ell_1(2)$ such that $E \setminus \{0\} \subset \hat{C}(\mathbb{R})$.*

In order to prove this proposition we need the following

Definition 11. A finite sequence $\tilde{e} = (e_1, \dots, e_n)$ in $C(\mathbb{R})$ is said to be *ε-Rademacher* ($\varepsilon \geqslant 0$) if there exist 2^n distinct points $t_1, \dots, t_{2^n}$ in $\mathbb{R}$ such that

- $\forall i \in \{1, \dots, n\}$, $\forall j \in \{1, \dots, 2^n\}$:

$$\|e_i\| = 1 \quad \text{and} \quad |e_i(t_j)| \in [1-\varepsilon, 1],$$

- $\forall \eta = (\eta_1, \dots, \eta_n)$ with $\eta_i = \pm 1$, $\exists j \in \{1, \dots, 2^n\}$ such that

$$\big(\operatorname{sign} e_1(t_j), \dots, \operatorname{sign} e_n(t_j)\big) = \eta.$$

If $\tilde{e}$ is ε-Rademacher for each $\varepsilon \gneq 0$ then $\tilde{e}$ is said to be *almost-Rademacher*. And, if $\tilde{e}$ is 0-Rademacher then $\tilde{e}$ is simply said *Rademacher*.

It is easy to prove the following

Lemma 12. *The sequence $\tilde{e} = (e_1, \dots, e_n)$ in $C(\mathbb{R})$ is isometrically equivalent to the unit basis of $\ell_1(n)$ if and only if $\tilde{e}$ is almost-Rademacher.*

Proof of Proposition 10. Let us suppose that there exists a two-dimensional subspace E of $C(\mathbb{R})$ with an almost-Rademacher basis $\tilde{e} = (e_1, e_2)$ such that $E \setminus \{0\} \subset \hat{C}(\mathbb{R})$. There are two cases:

- $\tilde{e}$ is Rademacher and then one of the four functions $-e_1$, e_1, $-e_2$ or e_2 has at least two maxima.
- $\tilde{e}$ is almost-Rademacher but not Rademacher. There exist t_1 and t_2 in $\mathbb{R}$ such that $e_i(t_i) = 1 = \max_{t \in \mathbb{R}} e_i(t)$, $i = 1, 2$. If $t_1 = t_2$ we define $e := e_1 - e_2$, if not $e := e_1 + e_2$. Since $\tilde{e}$ is almost-Rademacher,

for each $\varepsilon > 0$ there exists $t \in \mathbb{R}$ such that $e(t) \in [2-\varepsilon, 2[$. But, since e_1 and e_2 admit one and only one maximum: $\forall t \in \mathbb{R},\ e(t) < 2$. That means that the function e has no maximum and gives a contradiction which ends the proof. □

Let us note that we don't know if the set $\hat{C}(\mathbb{R})$ is n-lineable for $n > 3$, lineable or even spaceable. In the following section, we give a negative answer for vanishing functions.

3 The 2-lineability of $\hat{C}_0(\mathbb{R})$.

In this paragraph we will proof that there exists a two-dimensional vector subspace F of $C_0(\mathbb{R})$ such that $F \setminus \{0\} \subset \hat{C}_0(\mathbb{R})$ and that it is impossible to construct such a n-dimensional vector subspace for $n > 2$. Let us recall the notion of inclination.

Definition 13. Let P and Q be two closed subspaces of a Banach space $(X, \|.\|)$. The *inclination* of P on Q is defined by

$$(\widehat{P,Q}) := \inf\{d(x,Q) : x \in P,\ \|x\| = 1\}$$

where $d(x,Q) := \inf\{\|x-q\| : q \in Q\}$.

Remark 14. Clearly, if $P = \langle x \rangle$ and $Q = \langle y \rangle$ where x and y are linearly independent in X, then $(\widehat{P,Q})$ and $(\widehat{Q,P})$ are strictly positive. Moreover, if $(\widehat{P,Q}) = \delta > 0$ and $z = \alpha x + \beta y$ with $x \in P$, $y \in Q$ and $\|x\| = \|y\| = 1$ then $|\alpha| \leqslant \|z\|/\delta$.

Definition 15. A real-valued function x defined on a set K is said to be *alternating* if there exist t_1 and t_2 in K such that $f(t_1) < 0$ and

$f(t_2) > 0$. A set of functions is said to be alternating if every non zero function is alternating.

Proposition 16. *It is impossible to find an alternating two-dimensional vector subspace A of $C_0(\mathbb{R})$ such that $A \setminus \{0\} \subset \hat{C}_0(\mathbb{R})$.*

Proof. Let us suppose that there exist x and y two linearly independent functions such that $\langle x,y\rangle \setminus \{0\} \subset \hat{C}_0(\mathbb{R})$ and $\langle x,y\rangle \setminus \{0\}$ is alternating. Let us consider the set $Z := \{z = \alpha x + \beta y : \|z\| = 1\}$. By Remark 14, there exists $\delta > 0$ such that if $z = \alpha x + \beta y \in Z$ then α and β belong to $[-1/\delta, 1/\delta]$. Let us put, for every $z = \alpha x + \beta y \in Z$, $m_{\alpha\beta} := \inf\big\{(\alpha x + \beta y)(t) : t \in \mathbb{R}\big\}$ and $M_{\alpha\beta} := \sup\big\{(\alpha x + \beta y)(t) : t \in \mathbb{R}\big\}$. We have $\sup\{m_{\alpha\beta} : z = \alpha x + \beta y \in Z\} < 0$. Indeed, if not: $\exists (\alpha_n)_{n\geqslant 1}, (\beta_n)_{n\geqslant 1} \subset [-1/\delta, 1/\delta]$, $\forall \varepsilon > 0$, $\exists n_0 \geqslant 1$, $\forall n \geqslant n_0 : -\varepsilon \leqslant m_{\alpha_n\beta_n} \leqslant 0$. Up to a subsequence, we can assume that $\alpha_n \to \tilde{\alpha}$ and $\beta_n \to \tilde{\beta}$. Since, by Lemma 5, $m_{\alpha_n\beta_n} \to m_{\tilde{\alpha}\tilde{\beta}}$ we have $m_{\tilde{\alpha}\tilde{\beta}} = 0$. That means that $\tilde{z} = \tilde{\alpha}x + \tilde{\beta}y$ is positive which contradicts the fact that $\tilde{z}$ is alternating. In the same way, $\inf\{M_{\alpha\beta} : z = \alpha x + \beta y \in Z\} > 0$. Thus, let $N > 0$ be such that: $\forall z \in Z,\ m(z) < -N < 0 < N < M(z)$. Since x and y belong to $C_0(\mathbb{R})$ and since $z = \alpha x + \beta y \in Z$ implies α, $\beta \in [-1/\delta, 1/\delta]$, there exists $T > 0$ such that if $|t| \geqslant T$ and $z \in Z$ then $z(t) \in [-N, N]$. This implies that every $t \in \mathbb{R}$ such that $|t| \geqslant T$ is ignorable for $z \in Z$. So, the problem is reduced on $[-T, T]$: we have $\langle x,y\rangle \setminus \{0\} \subset \hat{C}([-T,T])$, which contradicts Theorem 6. □

Proposition 17. *Every n-dimensional ($n > 2$) vector space of functions contains an ($n-1$)-dimensional alternating subspace.*

In order to prove this proposition we need the following algebraic lemma.

Lemma 18. *Let V be an n-dimensional ($n \geqslant 2$) vector space of real-valued functions on a set K. Then there exist n points $(t_j)_{j=1}^n$ in K such that for every $(y_{ij}) \in \mathbb{R}^{n\times n}$, there exist n functions $(Y_i)_{i=1}^n$ of V such that $\forall i,j \in \{1,\ldots,n\} : Y_i(t_j) = y_{ij}$.*

Proof of Lemma 18. Clearly, if $\dim V = n$ then K contains at least n points.

- Let us begin by proving by induction that: if $\{X_i\}_{i=1}^n$ is a basis of V then there exist n points $\{t_i\}_{i=1}^n$ in K such that the n vectors $\big(X_1(t_j)\big)_{j=1}^n, \ldots, \big(X_n(t_j)\big)_{j=1}^n$ are linearly independent.

  For $n = 2$. Let us suppose, by contradiction, that for every t_1, t_2 in K the vectors $(X_1(t_1), X_1(t_2))$ and $(X_2(t_1), X_2(t_2))$ are linearly dependent. We can suppose that there exist t_0 in K and α in $\mathbb{R}$ such that $X_2(t_0) = \alpha X_1(t_0) \neq 0$ (if not, the assertion is trivial). Then, we have:

$$\forall t \in K,\ \exists \beta_t \in \mathbb{R}, \quad \big(\beta_t X_1(t_0), \beta_t X_1(t)\big) = \big(X_2(t_0), X_2(t)\big).$$

  The equality of the first components implies that for every t in K, $\beta_t = \alpha$ and then we have: $\forall t \in K,\ X_2(t) = \alpha X_1(t)$ which contradicts the fact that X_1 and X_2 are linearly independent in V.

  Let us suppose that the assertion is true for $n = k \geqslant 2$ and let us prove that it is longer true for $n = k+1$. Again, by contradiction, let us suppose that for every $\{t_j\}_{j=1}^{k+1} \subset K$, the vectors $\big(X_1(t_j)\big)_{j=1}^{k+1}, \ldots, \big(X_{k+1}(t_j)\big)_{j=1}^{k+1}$ are linearly dependent. Since the assertion is true for $n = k$, there exist $\{t_1, \ldots, t_k\} \subset K$ such that the span of the k vectors $\big(X_1(t_j)\big)_{j=1}^{k}, \ldots, \big(X_k(t_j)\big)_{j=1}^{k}$ is equal to $\mathbb{R}^k$.

Then, there exists an unique sequence $\{\alpha_i\}_{i=1}^k \subset \mathbb{R}$ such that

$$\Big(\sum_{i=1}^k \alpha_i X_i(t_j)\Big)_{j=1}^k = \big(X_{k+1}(t_j)\big)_{j=1}^k.$$

Indeed, since the rank of $(X_i(t_j))_{i,j=1}^k$ is equal to k, $(\alpha_i)_{i=1}^k$ is the unique solution of the system

$$\Big(\sum_{i=1}^k \beta_i X_i(t_j)\Big)_{j=1}^k = \big(X_{k+1}(t_j)\big)_{j=1}^k.$$

For every t in K the $k+1$ vectors

$$\Big(\big(X_1(t_j)\big)_{j=1}^k, X_1(t)\Big), \ldots, \Big(\big(X_{k+1}(t_j)\big)_{j=1}^k, X_{k+1}(t)\Big)$$

are linearly dependent. Then, for every t in K there exists $(\gamma_i)_{i=1}^k \subset \mathbb{R}$ such that

$$\Big(\big(\sum_{i=1}^k \gamma_i X_i(t_j)\big)_{j=1}^k, \sum_{i=1}^k \gamma_i X_i(t)\Big) = \big((X_{k+1}(t_j))_{j=1}^k, X_{k+1}(t)\big).$$

The equality of the k first components implies that $\{\gamma_i\}_{i=1}^k = \{\alpha_i\}_{i=1}^k$ and then we have: $\forall t \in K,\ X_{k+1}(t) = \sum_{i=1}^k \alpha_i X_i(t)$ which contradicts the fact that $(X_i)_{i=1}^{k+1}$ are linearly independent in V.

- Let us suppose that $\dim V = n$ and let us denote by $\{X_i\}_{i=1}^{k+1}$ a basis of V. By the previous step, there exists $(t_j)_{j=1}^n \in K$ such that the vectors $(X_1(t_j))_{j=1}^n, \ldots, (X_n(t_j))_{j=1}^n$ are linearly independent. Let us consider the matrix $(y_{ij}) \in \mathbb{R}^{n\times n}$. We have:

$$\forall j \in \{1,\ldots,n\},\ \exists \{\alpha_{ij}\}_{i=1}^n \subset \mathbb{R} : \Big(\sum_{i=1}^n \alpha_{ij} X_i(t_j)\Big)_{j=1}^n = (y_{ij})_{i=1}^n.$$

Then, the functions $\{Y_i\}_{i=1}^n \subset V$ defined by $Y_i = \sum_{j=1}^n \alpha_{ji} X_j$ are such that $Y_i(t_j) = y_{ij}$. □

Proof of Proposition 17. Let V be an n-dimensional vector space of functions on K and let us suppose that every subspace W of V of dimension $n-1$ is not alternating. Then there exists a positive function Y on K. Let us choose n points in K and consider the vector $(y_i)_{i=1}^n$ of the values of Y on these points. Clearly, the orthogonal complement of this vector in $\mathbb{R}^n$ is an alternating vector subspace of $\mathbb{R}^n$ of dimension $n-1$. Let $(y_{1j})_{j=1}^n, \dots, (y_{(n-1)j})_{j=1}^n$ be a basis of this subspace of $\mathbb{R}^n$. By Lemma 18, there exist n points $\{t_j\}_{j=1}^n \subset K$ and n functions $\{Y_i\}_{i=1}^n \subset V$ such that $Y_i(t_j) = y_{ij}$. Clearly, $W = \langle Y_i \rangle_{i=1}^n$ is an alternating subspace of V of dimension $n-1$. □

We can now easily prove the announced

Theorem 19. $\lambda(\hat{C}_0(\mathbb{R})) = 2$.

Proof. We have $\langle \sin, 1-\cos \rangle \setminus \{0\} \subset \hat{C}([0, 2\pi[)$ and then, as in the proof of Theorem 9, we have that $\hat{C}_0(\mathbb{R})$ is 2-lineable. The fact that $\hat{C}_0(\mathbb{R})$ is not n-lineable for $n > 2$ is a straightforward consequence of Propositions 16 and 17. □

If we denote by $C_L(\mathbb{R})$ the set of functions defined on $\mathbb{R}$ such that the limits $\lim_{t \to -\infty} f(t)$ and $\lim_{t \to +\infty} f(t)$ exist, we have the following corollary of Theorem 19:

Corollary 20. $\lambda(\hat{C}_L(\mathbb{R})) = 2$.

Remark 21. Using the very non linearity of $\|\hat{C}[0,1]\|$ (see Remark 7) instead of Theorem 6 in the proof of Proposition 16, we can prove that: it is impossible to find an alternating two-dimensional vector subspace A of $C_0(\mathbb{R})$ such that $A \setminus \{0\} \subset \|\hat{C}_0(\mathbb{R})\|$ (where $\|\hat{C}_0(\mathbb{R})\|$ is the subset of $C_0(\mathbb{R})$ which attains their supremum norm at a unique point). So, Proposition 17 implies: $\lambda(\|\hat{C}_0(\mathbb{R})\|) \leqslant 2$. We don't know if this set is 2-lineable or very non linear.

Surprisingly, the corresponding result for the space of convergent sequences is different: the set $\hat{c}_0$ of vanishing real sequences with an unique maximum is very non linear.

Proposition 22. $\lambda(\hat{c}_0) = 1$.

Proof. Let us suppose, by contradiction, that there exist two linearly independent elements $x = (x_n)_{n\geqslant 1}$ and $y = (y_n)_{n\geqslant 1}$ of c_0 such that for every (α, β) in $\mathbb{R}^2 \setminus \{(0,0)\}$, $\alpha x + \beta y$ admits one and only one maximum. Without loss of generality we can suppose that $\max_{i\geqslant 1} x_i = x_{i_0} = 1$, $y_{i_0} = 0$ and that there exists $j_0 \neq i_0$ such that $y_{j_0} > 0$. Let $\lambda_{j_0} \in \mathbb{R}$ be such that $x_{j_0} + \lambda_{j_0} y_{j_0} = 1$ and let us consider $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < 1/(1+\lambda_{j_0})$. Since the sequences x and y converge to 0: $\exists N > j_0,\ \forall i \geqslant N,\ \max\{|x_i|, |y_i|\} < \varepsilon$. Let us consider $\{y_{i_k}\}_{k=1}^m \subset \{y_i\}_{i=1}^{N-1}$ such that $\forall k \in \{1,\ldots,m\}$, $y_{i_k} > 0$ and $\{\lambda_k\}_{k=1}^m \subset \mathbb{R}$ such that $x_{i_k} + \lambda_k y_{i_k} = 1$. So, $\lambda_0 := \min\{\lambda_k\}_{k=1}^m > 0$. Let us define the sequence $z := x + \lambda_0 y$. It is such that $\max_{i\geqslant 1} z_i = 1$, $z_{i_0} = x_{i_0} = 1$ and $\forall k \in \{1,\ldots,m\}$ such that $\lambda_k = \lambda_0$: $z_{i_k} = 1$. Then z has at least two maxima, which is a contradiction. $\square$

The following proposition is proved in [14]. We give here a proof of the same result based on the proof of Proposition 22.

Proposition 23. *Let $L \subset c_0$ be a subspace with $\dim L = n \in \mathbb{N} \setminus \{0\}$. Then there exists $x \in L$ such that $\|x\|_\infty = 1$ and $\big|\{i : |x_i| = 1\}\big| \geqslant n$.*

In particular, this proposition implies that the subset $\|\hat{c}_0\|$ of c_0 of sequences which attain their norm at a unique point is very non linear:

Corollary 24. $\lambda(\|\hat{c}_0\|) = 1$.

Since the sup-norm of c_0 is Gâteaux-differentiable at x if and only if $t \mapsto |x(t)|$ attains its supremum over $\mathbb{N}$ at a single point t_0 and $|x(t_0)| > \sup\{|x(t)| : t \in \mathbb{N} \setminus \{t_0\}\}$ (cfr [5]), we have

Corollary 25. *The set of points of Frechet-differentiability of the supremum norm of c_0 is very non linear.*

Proof Proposition 23. Let $\{x^1, \ldots, x^n\}$ be a basis of L. Let us proceed by induction on the dimension n of L.

The case $n = 1$ is trivial.

The case $n = 2$. Let us suppose, by contradiction, that for every $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $\alpha x^1 + \beta x^2$ attains its norm at a unique point. Without loss of generality we can suppose that $\|x^1\|_\infty = x^1_{i_0} = 1$, $x^2_{i_0} = 0$ and that there exists $j_0 \neq i_0$ such that $x^2_{j_0} \neq 0$. Let us define the positive real number λ_{j_0} such that $x^1_{j_0} + \lambda_{j_0} x^2_{j_0} = \operatorname{sign} x^2_{j_0}$ and consider $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < 1/(1 + \lambda_{j_0})$. Since the sequences x^1 and x^2 converge to 0: $\exists N > j_0$, $\forall i \geqslant N : \max\{|x^1_i|, |x^2_i|\} < \varepsilon$. For $i \in \{1, \ldots, N-1\}$ and such that $x^2_i \neq 0$, let us define $\lambda_i \in \mathbb{R}$ such that $x^1_i + \lambda_i x^2_i = \operatorname{sign} x^2_i$. Let us consider $\Lambda_0 = \min\{\lambda_i\} > 0$ and the sequence $w^0 = x^1 + \Lambda_0 x^2$. We have $\|w^0\|_\infty = 1$, $w^0_{i_0} = x^1_{i_0} = 1$ and for all $i \in \{1, \ldots, N-1\}$ such that $\lambda_i = \Lambda_0$: $w^0_i = \operatorname{sign} x^2_i$. Then w^0 attains its norm at at least two distinct points, a contradiction.

The case $n = 3$. Let us suppose, by contradiction, that the proposition is false for $n = 3$. Thus, for w^0 defined in the previous step, there exists only one $i_1 \in \mathbb{N}$ such that $\lambda_{i_1} = \Lambda_0$. Without loss of generality we can suppose that $x^3_{i_0} = x^3_{i_1} = 0$ and that there exists $j_1 \notin \{i_0, i_1\}$ such that $x^3_{j_1} \neq 0$. Let us define the positive real number λ_{j_1} such that $w^0_{j_1} + \lambda_{j_1} x^3_{j_1} = \operatorname{sign} x^3_{j_1}$ and consider $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < 1/(1 + \lambda_{j_1})$. Since the sequences w^0 and x^3 converge to 0: $\exists N > j_1,\ \forall i \geqslant N:$ $\max\{|w^0_i|, |x^3_i|\} < \varepsilon$. For $i \in \{1, \dots, N-1\}$ and such that $x^3_i \neq 0$, let us define $\lambda_i \in \mathbb{R}$ such that $w^0_i + \lambda_i x^3_i = \operatorname{sign} x^3_i$. Let us consider $\Lambda_1 = \min\{\lambda_i\} > 0$ and the sequence $w^1 = w^0 + \Lambda_1 x^3$. We have $\|w^1\|_\infty = 1$, $w^1_{i_0} = w^0_{i_0} = 1$, $w^1_{i_1} = w^0_{i_1} = \operatorname{sign} x^2_{i_1}$ and for all $i \in \{1, \dots, N-1\}$ such that $\lambda_i = \Lambda_1$: $w^1_i = \operatorname{sign} x^3_i$. Then w^1 attains its norm at at least three distinct points, a contradiction.

We can now use the same idea to perform the step $n = 4$ and so on. $\square$

Let us remark that, according to some minor modifications in this proof, we can get the following improvement of Proposition 22

Corollary 26. *Let $L \subset c_0$ be a subspace with $\dim L = n \in \mathbb{N} \setminus \{0\}$. Then there exists $x \in L$ such that $\|x\|_\infty = 1$ and $\big|\{i : x_i = 1\}\big| \geqslant n$.*

4 The spaceability of $\widetilde{CB}(\mathbb{R})$.

Let us consider the set $\widetilde{CB}(\mathbb{R})$ (resp. $\|\widetilde{CB}(\mathbb{R})\|$) of continuous and bounded real-valued functions defined on $\mathbb{R}$ which do not attain their supremum (resp. their supremum norm).

Theorem 27. *$\widetilde{CB}(\mathbb{R})$ and $\|\widetilde{CB}(\mathbb{R})\|$ are spaceable.*

By linear interpolations and symmetrisation, this theorem is a straightforward corollary of the corresponding following result concerning sequences:

Proposition 28. *$\widetilde{\ell_\infty}$ and $\|\widetilde{\ell_\infty}\|$ are spaceable.*

Proof. Let us consider the set of sequences $(s_n)_{n\in\mathbb{N}} \subset \ell_\infty$ defined by:

$$\forall n \in \mathbb{N}, \quad s_n = (s_{n,i})_{i\geqslant 1} = (\underbrace{1,1,\ldots,1}_{2^n \text{ times}},\underbrace{-1,-1,\ldots,-1}_{2^n \text{ times}},\underbrace{1,1,\ldots,1}_{2^n \text{ times}},\ldots).$$

And with these sequences let us construct the following one's:

$$\forall n \in \mathbb{N}, \quad e_n = \big((1-1/2^i)s_{n,i}\big)_{i\geqslant 1}.$$

For every $N \in \mathbb{N}$, we have:

$$\Big\|\sum_{n=0}^{N} \alpha_n e_n\Big\|_\infty = \sum_{n=0}^{N} |\alpha_n|$$

which proves that $(e_n)_{n\in\mathbb{N}}$ is a basic sequence equivalent to the canonical basis of ℓ_1. Obviously, we have:

$$\forall i \geqslant 1, \quad \Big|\Big(\sum_{n=0}^{+\infty} \alpha_n e_n\Big)_i\Big| < \Big\|\sum_{n=0}^{+\infty} \alpha_n e_n\Big\|_\infty = \sum_{n=0}^{+\infty} |\alpha_n| = \sup_{i\geqslant 1}\Big(\sum_{n=0}^{+\infty} \alpha_n e_n\Big)_i.$$

This proves that $E = \langle e_n \rangle_{n\in\mathbb{N}}$ is an infinite dimensional closed vector subspace of ℓ_∞ such that $E \setminus \{0\} \subset \widetilde{\ell_\infty} \cap \|\widetilde{\ell_\infty}\|$. □

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