

Preprint #12
October 12, 2002

Some remarks on M -ideals and Strong Proximality

Catherine FINET
Lucas QUARTA



Université de Mons-Hainaut
Institut de Mathématique

Phone: +32 65 37 35 07 — Fax: +32 65 37 33 18

Web: <http://www.umh.ac.be/math/institut>

Some remarks on M -ideals and Strong Proximality

Catherine Finet*

Lucas Quarta*

catherine.finet@umh.ac.be

lucas.quarta@umh.ac.be

Université de Mons-Hainaut,
Institut de Mathématique,
“Le Pentagone”
Avenue du Champ de Mars, 6
7000 Mons, Belgium

Abstract. We prove that every M -ideal is strongly proximal and that, for any Banach space X , $K(X, c_0)$ is an M -ideal in $L(X, \ell^\infty)$.

Mathematics Subject Classification: 41A50, 41A65, 46B20.

Keywords and phrases: Proximality, M -ideals, n -ball property, $1\frac{1}{2}$ -ball property, compact operators.

*The authors want to thank the “Banque Nationale de Belgique” for the grant they got.

1 Introduction

Many papers [2, 10, 11, 12, 13, 14, 16] are concerned with the phenomenon that for certain Banach spaces X and Y , the Banach space $K(X, Y)$ of compact linear operators from X to Y is an M -ideal in $L(X, Y)$, the Banach space of bounded linear operators from X to Y . This special attention is due to the facts that, for example, when this happens:

- every element of $L(X, Y)$ has a best compact approximant (cfr. **Theorem A** and **Theorem B**),
- one has the uniqueness of Hahn-Banach extensions from $K(X, Y)$ to $L(X, Y)$ [14].

This article extends some results of [2]. The authors proved that if X is a Banach space with an unconditional shrinking basis then $K(X, c_0)$ is proximal in $L(X, \ell^\infty)$. Here, we prove that every M -ideal is strongly proximal. We also show that $K(X, c_0)$ is an M -ideal in $L(X, \ell^\infty)$. This implies that, without any assumption on X , $K(X, c_0)$ is strongly proximal in $L(X, \ell^\infty)$.

Let us recall the notions mentioned above.

Let J be a closed subspace of a Banach space X . For $x \in X$, let

$$P_J(x) := \{j \in J : \|x - j\| = d(x, J)\}$$

where $d(x, J) := \inf\{\|x - j\| : j \in J\}$. The subspace J is said to be *proximal* in X , if for each x in X , the set $P_J(x)$ is non-empty. An element of $P_J(x)$ is called a *best approximant* of x in J . Every closed subspace of a finite dimensional space or of a uniformly convex space (and then of a reflexive space) is clearly proximal. It is known that a Banach space X is reflexive if and only if every closed hyperplane is proximal in X . If J is a proximal subspace of finite codimension in X then $J^\perp := \{x^* \in X^* : x^*|_J \equiv 0\}$ is contained in $\text{NA}(X)$, the subset of X^* consisting of all norm attaining functionals on X [5]. The converse fails in general (see [15]) but it is true for c_0 and its subspaces [6].

When each “nearly best approximant” of x in J is necessarily close to an actual best approximant one says that J is *strongly proximal* in X . More precisely, for $\delta > 0$ let $P_J(x, \delta) := \{j \in J : \|x - j\| < d(x, J) + \delta\}$, then J is strongly proximal in X if and only if J is proximal in X and for each x in X and each $\varepsilon > 0$ there is $\delta > 0$ such that $d(j, P_J(x)) < \varepsilon$ for all $j \in P_J(x, \delta)$. It is easy to prove that every closed subspace of a finite dimensional space is strongly proximal. But in general proximality does not imply strong proximality. For example, every proximal hyperplane is strongly proximal if and only if $\text{NA}_1(X) := \text{NA}(X) \cap S(X^*)$ coincides with the set $S := \{x^* \in S(X^*) : \|\cdot\|_{X^*} \text{ is SSD at } x^*\}$ [7]. A norm $\|\cdot\|$ on X is said to be *strongly subdifferentiable (SSD)* at x if the one-side limit $\lim_{t \rightarrow 0^+} \frac{1}{t}(\|x + th\| - \|x\|)$ exists uniformly in $h \in S(X)$. In [4], it is shown that $S \subset \text{NA}_1(X)$. Then, in a reflexive infinite dimensional space with a dual norm which is not everywhere SSD, there exists a hyperplane which is not strongly proximal. In ℓ_1 , there exist non strongly proximal hyperplanes since the canonical sup-norm on ℓ^∞ is not everywhere SSD (see [3]). In [8], the authors proved that a finite codimensional subspace J of $K(\ell^2)$ is strongly proximal if and only if J^\perp is contained in $\text{NA}(K(\ell^2))$.

A closed subspace J in X is said to be an *M-ideal* in X if and only if there exists a linear projection P from X^* to J^\perp such that:

$$\forall x^* \in X^*, \quad \|x^*\| = \|Px^*\| + \|x^* - Px^*\|.$$

This notion is due to Alfsen and Effros [1] and studied in detail in [9].

Examples of *M-ideals* are [9]: c_0 in its bidual ℓ^∞ , $K(H)$ in $L(H)$ where H is an Hilbert space, $K(X, c_0)$ in $L(X, c_0)$ for every Banach space X , or $K(\ell^p, \ell^q)$ in $L(\ell^p, \ell^q)$ for $1 < p, q < +\infty$, while it is not true for $K(X)$ in $L(X)$ if $X = \ell^\infty$ or $X = L^p(0, 1)$, $p \in [1, +\infty] \setminus \{2\}$.

In [12], it is shown that $K(L^p, \ell^p)$ is an *M-ideal* in $L(L^p, \ell^p)$ if $1 < p \leq 2$ and it is not true if $p > 2$. In [10], it is proved for $2 \leq p < +\infty$ and subspaces X of quotients of L^p with a 1-unconditional finite dimensional Schauder decomposition that $K(X, \ell^p)$ is an *M-ideal* in $L(X, \ell^p)$.

In [1], the authors also gave the following equivalent condition for J to be an M -ideal in X which avoids mentioning the dual space X^* .

Theorem A. *For a closed subspace J of a Banach space X , the following assertions are equivalent:*

1. J is an M -ideal,

2. *The n -ball property.*

For all $n \in \mathbb{N}$ and all families $(B(x_i, r_i))_{i=1}^n$ of n closed balls satisfying $B(x_i, r_i) \cap J \neq \emptyset$ for all $i = 1, \dots, n$ and $\bigcap_{i=1}^n B(x_i, r_i) \neq \emptyset$ the conclusion $\bigcap_{i=1}^n B(x_i, r_i + \varepsilon) \cap J \neq \emptyset$ for all $\varepsilon > 0$ holds.

3. *The [restricted] 3-ball property.*

For all $j_1, j_2, j_3 \in B(J)$, all $x \in B(X)$ and all $\varepsilon > 0$ there is $j \in J$ satisfying: $\|x + j_i - j\| \leq 1 + \varepsilon$ for $i = 1, 2, 3$.

Following [16], we say that a closed subspace J of a Banach space X has the $1\frac{1}{2}$ -ball property in X if the conditions

$$x \in X, j \in J, \quad \|x - j\| \leq r_1 + r_2 \quad \text{and} \quad B(x, r_2) \cap J \neq \emptyset$$

implies that

$$B(j, r_1) \cap B(x, r_2) \cap J \neq \emptyset.$$

This is equivalent to requiring the (strict) 2-ball property subject to the restriction that one of the centers lies in J :

$$\text{if } x \in X, j \in J, B(j, r_1) \cap B(x, r_2) \neq \emptyset \text{ and } B(x, r_2) \cap J \neq \emptyset$$

$$\text{then } B(j, r_1) \cap B(x, r_2) \cap J \neq \emptyset.$$

Let us note that the $1\frac{1}{2}$ -ball property is not a sufficient condition to be an M -ideal: $K(\ell^1)$ has the $1\frac{1}{2}$ -ball property in $L(\ell^1)$ but it is not an M -ideal (see [9, 16]).

Theorem B ([16]). *Let J be a closed subspace of a Banach space X . If J has the $1\frac{1}{2}$ -ball property in X then J is proximal in X .*

According to **Theorem A** and **Theorem B**, we get

Corollary. *Every M -ideal is proximal.*

2 Results

As mentioned before, the $1\frac{1}{2}$ -ball property is a sufficient condition for proximality. Here we prove that this property implies strong proximality.

Theorem 1. *Let J be a closed subspace of a Banach space X . If J has the $1\frac{1}{2}$ -ball property in X then J is strongly proximal in X .*

Proof. The proof follows the ideas of the proof of **Theorem B** in [16]. Let $x \in X$ be such that $d := d(x, J) > 0$ and $\varepsilon > 0$ be fixed. We want to prove that there exists $\delta > 0$ such that:

$$[\forall j \in J, \|x - j\| < d + \delta] \Rightarrow [\exists j' \in J, \|j - j'\| < \varepsilon \text{ and } \|x - j'\| = d].$$

Let us take $\delta = \varepsilon$ and let $j_1 \in J$ be such that $\|x - j_1\| < d + \delta = (d + \delta/2) + \delta/2$. By definition of d , $B(x, d + \delta/2) \cap J \neq \emptyset$ and then by the $1\frac{1}{2}$ -ball property, we have

$$\exists j_2 \in J, \quad j_2 \in B\left(j_1, \frac{\delta}{2}\right) \cap B\left(x, d + \frac{\delta}{2}\right).$$

We have now $\|x - j_2\| \leq (d + \delta/2) = (d + \delta/4) + \delta/4$ and by the $1\frac{1}{2}$ -ball property, we have:

$$\exists j_3 \in J, \quad j_3 \in B\left(j_2, \frac{\delta}{4}\right) \cap B\left(x, d + \frac{\delta}{4}\right).$$

So, inductively, we construct a sequence $(j_n)_{n \geq 1} \subset J$ such that: for all $n \geq 1$,

$$\|x - j_n\| \leq d + \frac{\delta}{2^{n-1}} \tag{1}$$

$$\|j_n - j_{n+1}\| \leq \frac{\delta}{2^n} \tag{2}$$

By (2), $(j_n)_{n \geq 1}$ is a Cauchy sequence in J . Since J is closed, there exists $j' \in J$ such that $j' = \lim_{n \rightarrow \infty} j_n$. By (1), we have $\|x - j'\| = d$. By (2) again,

$$\begin{aligned} \forall n \geq 1, \quad \|j_1 - j_n\| &\leq \|j_1 - j_2\| + \|j_2 - j_3\| + \cdots + \|j_{n-1} - j_n\| \\ &\leq \delta \sum_{n=1}^{+\infty} \frac{1}{2^n} = \delta. \end{aligned}$$

Then, $\|j_1 - j'\| \leq \delta = \varepsilon$. □

By **Theorem A** and **Theorem 1**, we have the following

Corollary 2. *Every M -ideal is strongly proximal.*

Following ideas described in [9], we prove

Proposition 3. *For every Banach space X , $K(X, c_0)$ is an M -ideal in $L(X, \ell^\infty)$.*

Proof. By **Theorem A**, it suffices to prove that $K(X, c_0)$ satisfies the 3-ball property in $L(X, \ell^\infty)$. Let $S \in B(L(X, \ell^\infty))$, $T_i \in B(K(X, c_0))$ ($i = 1, 2, 3$) and $\varepsilon > 0$. We want to find $T \in K(X, c_0)$ such that:

$$\|S + T_i - T\| \leq 1 + \varepsilon \quad \text{for } i = 1, 2, 3.$$

For $a = (a_j)_{j \geq 1} \in \ell^\infty$, let us put $P_n(a) = (a_1, a_2, \dots, a_n, 0, \dots)$. Since $T_i \in K(X, c_0)$ ($i = 1, 2, 3$), we have:

$$\|P_n T_i - T_i\| < \varepsilon \quad \text{for } i = 1, 2, 3 \text{ and } n \text{ big enough.}$$

Moreover,

$$\|P_n T_i + (\mathbb{1}_{\ell^\infty} - P_n)S\| \leq 1 \quad (i = 1, 2, 3),$$

since $\|P_n T_i + (\mathbb{1}_{\ell^\infty} - P_n)S\| \leq \max\{\|P_n T_i\|, \|(\mathbb{1}_{\ell^\infty} - P_n)S\|\}$.

If we take $T = P_n S$ (for n big enough), then for $i = 1, 2, 3$:

$$\|T_i + S - P_n S\| \leq \|T_i - P_n T_i\| + \|P_n T_i + (\mathbb{1}_{\ell^\infty} - P_n)S\| \leq \varepsilon + 1. \quad \square$$

By **Corollary 2**, we then have

Corollary 4. *For every Banach space X , $K(X, c_0)$ is strongly proximal in $L(X, \ell^\infty)$.*

Remark 5. It is clear from the proof of [Proposition 3](#) that if (K_α) is a net of compact operators on Y satisfying:

$$\begin{aligned} &K_\alpha \rightarrow \mathbb{1}_Y \text{ in norm and} \\ &\text{for all } U, W \text{ in } L(X, Y), \\ &\limsup_{\alpha} \|K_\alpha U + (\mathbb{1}_Y - K_\alpha)W\| \leq \max \{\|U\|, \|W\|\} \end{aligned}$$

then $K(X, Y)$ is an M -ideal in $L(X, Y)$.

References

- [1] E.M. ALFSEN, E.G. EFFROS, Structure in real Banach spaces. Part I and II, *Ann. of Math.* **96** (1972), 98–173.
- [2] C.M. CHO, W.S. ROH, Proximality of certain spaces of compact operators, *Bull. Korean Math. Soc.* **38** (2001), No 1, 65–69.
- [3] C. FRANCHETTI, Lipschitz maps and the geometry of the unit ball in normed spaces, *Arch. Math.* **46** (1986), 76–84.
- [4] C. FRANCHETTI, R. PAYÁ, Banach spaces with strongly subdifferentiable norm, *Boll. Uni. Mat. Ital.* **VII**. B (1993), 45–70.
- [5] A.L. GARKAVI, On the best approximation by elements of infinite dimensional subspaces of a certain class, *Math. Sb.* **62** (1963), 104–120.
- [6] G. GODEFROY, V. INDUMATHI, Proximality in subspaces of c_0 , *J. Approx. Theory* **101** (1999), 175–181.
- [7] G. GODEFROY, V. INDUMATHI, Strong Proximality and Polyhedral spaces, *Revista Math. Complut.* **14** (2001), No 1, 105–125.
- [8] G. GODEFROY, V. INDUMATHI, F. LUST-PIQUARD, Strong Subdifferentiability of convex functionals and proximality, *J. Approx. Theory* **116** (2002), No 2, 397–415.

- [9] P. HARMAND, D. WERNER, W. WERNER, *M*-ideals in Banach spaces and Banach Algebras, Lecture Notes in Math. **1547**, Springer, Berlin-Heidelberg-New York, 1993.
- [10] K. JOHN, D. WERNER, *M*-ideals of compact operators into ℓ^p , Czech. Math. Journal **50 (125)** (2000), 51–57.
- [11] N.J. KALTON, *M*-ideals of compact operators, Illinois J. Math. **37** (1993), 147–169.
- [12] N.J. KALTON, D. WERNER, Property (*M*), *M*-ideals and almost isometric structure of Banach spaces, J. Reine Angew. Math. **461** (1995), 137–178.
- [13] A. LIMA, E. OJA, T.S.S.R. RAO, D. WERNER, Geometry of Operator Spaces, Mich. Math. J. **41** (1994) No 3, 473–490.
- [14] R.R. PHELPS, Uniqueness of Hahn-Banach extensions and unique best approximation, Trans. Amer. Math. Soc. **95** (1960), 238–255.
- [15] R.R. PHELPS, Chebychev subspaces of finite codimension in $C(X)$, Pacific J. Math. **13** (1963), 647–655.
- [16] D.T. YOST, Best Approximation and Intersections of balls in Banach spaces, Bull. Austral. Math. Soc. **20** (1979), 258–300.

Recent preprints

[4] Catherine FINET & Lucas QUARTA & Christophe TROESTLER, *Vector-valued Variational Principles*, January 19, 2001.

[5] Gilles GODEFROY, *Montons les degrés*, 8 mars 2001.

[6] Paul VAN PRAAG, *Quaternions as reflexive skew fields*, 22 mars 2001.

[7] Maurice BOFFA, *Théorie des ensembles et dualité*, 17 avril 2001.

[8] Paul VAN PRAAG, *Pedro Nuñez, Simon Stevin, et le plus grand commun diviseur des polynômes*, 17 avril 2001.

[9] TROESTLER C., *Equivalence of BSS scalar- and vector-recursion*, June 4, 2001.

[10] Christian MICHAUX & Adem OZTURK, *Quantifier Elimination following Muchnik*, April 11, 2002.

[11] Christian MICHAUX & Cédric RIVIÈRE, *Axiomatisation à la Pierce-Pillay pour les corps ordonnés différentiellement clos*, 4 juillet 2002.

You can find more informations as well as download the preprints of the *Institut de Mathématique* on the web site: <http://www.umh.ac.be/math/preprints/>. Printed copies are available upon request by writing to:

Institut de Mathématique
Université de Mons-Hainaut
« Le Pentagone », 6 av. du champ de Mars
7000 Mons, Belgium