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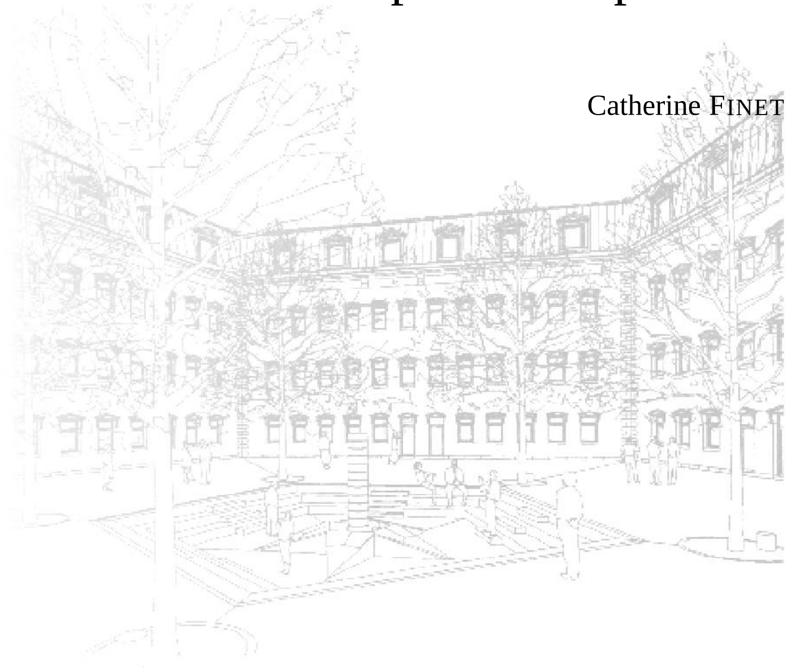

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Numerical Ranges of Some Composition Operators

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Abstract. This paper is a short survey on the numerical range of some composition operators. The first part is devoted to composition operators on the Hilbert Hardy space H^2 on the unit disk. The results are due to P. Bourdon, J. Shapiro and V. Matache.

In the second part we study the numerical range of composition operators on the Hilbert space $\mathscr{H}^2$ of Dirichlet series. These results are due to H. Queffélec and the author.

The third part is devoted to compactness connected with fixed points in the setting of H^2 and $\mathscr{H}^2$. These results are due to H. Queffélec and the author.

Introduction

The setting is the following : a set X, a Banach space of a collection of functions on X. If ϕ maps X into itself, the composition operator C_ϕ is defined by $(C_\phi f)(x) = f(\phi(x))$, for x in X and functions f in the Banach space. The function ϕ is called the *symbol* of the composition operator.

The numerical range of a linear bounded operator T on a Hilbert space H is the set

$$W(T) = \left\{\langle Tf, f\rangle,\ f \in H,\ \|f\| = 1\right\}.$$

The numerical range has the following properties :

a) it contains every eigenvalue of T (obvious),

b) it lies in the disk $\{|w| \leq \|T\|\}$ (obvious),

c) its closure contains the spectrum of T (easy),

d) it is convex (Toeplitz-Hausdorff theorem), therefore Lebesgue measurable,

e) it is even a Borel set ([1]),

f) for compact T, it is closed if and only if it contains 0 ([6]).

We describe the shape of the numerical range of some composition operators C_ϕ. Of course this is clearly connected with the functional properties of ϕ.

In the first part we work in the Hilbert Hardy space H^2 of analytic functions $f(z) = \sum_0^\infty \hat{f}(n) z^n$ on the open unit disk $\mathbb{D}$ with square-summable coefficients : $\|f\|^2 = \sum_0^\infty |\hat{f}(n)|^2 < \infty$ and reproducing kernel K_a ($a \in \mathbb{D}$), i.e. $f(a) = \langle f, K_a\rangle$ and $K_a(z) = 1/(1-\bar{a}z)$.

In the second part we work with the Hilbert Hardy-Dirichlet space $\mathscr{H}^2$ of analytic functions f admitting a Dirichlet series expansion $f(s) = \sum_1^\infty a_n n^{-s}$

with square-summable coefficients $\|f\|^2 = \sum_1^\infty |a_n|^2 < \infty$. As we will see the two situations are very different.

The last part is essentially devoted to the study of the connection between compactness and fixed points. This part has not been published.

We only give the proofs not published.

1 Composition operators on H^2

By the famous Littlewood's subordination principle [14], each holomorphic selfmap ϕ of $\mathbb{D}$ induces on H^2 a bounded composition operator C_ϕ with

$$\|C_\phi\| \leq \left(\frac{1+|\phi(0)|}{1-|\phi(0)|}\right)^{1/2}.$$

Let us also mention if ϕ is inner then we have the equality :

$$\|C_\phi\| = \left(\frac{1+|\phi(0)|}{1-|\phi(0)|}\right)^{1/2}.$$

1.1 Numerical range for monomial symbols [12]

1. $\phi(z) = z$
   $W(C_\phi) = \{1\}$.
2. $\phi(z) = r$
   - If $r = 0$, then $W(C_\phi) = [-1, 1]$,
   - If $r \neq 0$, $|r| < 1$, then $W(C_\phi)$ is the closed elliptic disk whose boundary is the ellipse of foci 0 and 1 having horizontal axis of length $1/\sqrt{1-r^2}$.
3. $\phi(z) = \omega z$, $|\omega| = 1$, $\omega \neq 1$ (ϕ is a ω-rotation).

- If ω is a primitive root of unity of order $n \geq 2$ then $W(C_\phi)$ is the convex hull of all the n-th roots of unity. In particular, when $n = 2$, $W(C_\phi)$ is just the closed segment $[-1, 1]$.
- If ω is not a root of unity then $W(C_\phi)$ is the union of $\mathbb{D}$ and the set $\{\omega^n, n \geq 0\}$.

4. $\phi(z) = rz$, $|r| < 1$ (ϕ is a dilatation, more precisely, an r-dilatation)
   - $r > 0$ (ϕ is a positive dilatation), $W(C_\phi) =]0, 1]$
   - $r \leq 0$, $W(C_\phi) = [r, 1]$
   - $r \notin \mathbb{R}$, $W(C_\phi)$ is a closed polygonal region, whose vertices form a finite subset of the set $\{r^n,\ n \geq 0\}$.

5. $\phi(z) = cz^k$, $|c| = 1$, $k \geq 2$
   $W(C_\phi) = \mathbb{D} \cup \{1\}$.

6. $\phi(z) = cz^k$, $|c| < 1$, $c \neq 0, k \geq 2$
   $W(C_\phi)$ is the convex hull of the point 1 and the disk centered at 0 with radius $1/\sqrt{t}$ (see [12] for the value of t).

1.2 Numerical range for automorphism symbols [12]

Automorphisms of the unit disk (one-to-one analytic maps of $\mathbb{D}$ onto itself) are the mappings

$$\phi(z) = \lambda \frac{a - z}{1 - \bar{a}z}, \qquad |\lambda| = 1,\ |a| < 1.$$

The classification of the automorphisms is the following :

- Elliptic automorphisms of $\mathbb{D}$ are conjugate to rotations.
- Hyperbolic automorphisms of $\mathbb{D}$ are conjugate to positive dilatations.
- Parabolic automorphisms of $\mathbb{D}$ are conjugate to translations.
- All other are conjugate to complex dilatations.

Let us mention

Theorem 1 ([7]) *Let ϕ be any holomorphic selfmap of $\mathbb{D}$, the following are equivalent:*

1. *C_ϕ is invertible,*
2. *C_ϕ is Fredholm,*
3. *ϕ is an automorphism of the unit disk.*

1. ϕ is an automorphism of $\mathbb{D}$ that is either parabolic or hyperbolic.
   $W(C_\phi)$ is a disk centered at the origin.
   $W(C_\phi)$ is either open or closed.

2. ϕ is an elliptic automorphism of $\mathbb{D}$.
   Then ϕ is conjugate to a rotation, $\phi = \tau^{-1} \circ \varphi \circ \tau$ where φ is a ω-rotation.
   - If ω is not a root of unity, $\overline{W(C_\phi)}$ is a disk centered at the origin.
   - If ω is a primitive root of order 2, $\overline{W(C_\phi)}$ is an elliptic disk with foci at ± 1.
   - If ω is a primitive root of order $n > 2$, the situation is not well-known.

1.3 Zero containement [4]

Let us recall that the numerical range of a compact operator is closed if an only if it contains the origin [6]. As we saw before the positive dilatations induce a class of compact composition operators with non closed numerical range ($W(C_\phi) =]0,1]$).

Theorem 2 *If ϕ is any holomorphic selfmap of $\mathbb{D}$ that is not the identity then $0 \in \overline{W(C_\phi)}$.*

Let us now consider the following question: *For which ϕ does int $W(C_\phi)$ contain the origin ?*

The case ϕ is constant is solved in Section 1.1.

Theorem 3 *Let ϕ be a non constant holomorphic selfmap of $\mathbb{D}$. If ϕ is not one-to-one then* $0 \in \operatorname{int} W(C_\phi)$.

We now consider this question for maps that fix *the origin*. The case of dilatations has already been treated before. Let us consider the non dilatation case.

Theorem 4 *Let ϕ be a non constant holomorphic selfmap of $\mathbb{D}$. If ϕ is not a dilatation and $\phi(0) = 0$ then* $0 \in \operatorname{int} W(C_\phi)$.

Let us mention as a consequence of this theorem that when ϕ is not a positive dilatation with $\phi(0) = 0$, if C_ϕ is compact then $W(C_\phi)$ is closed. What's happening for maps that fix a non-zero point ? The theorem carries over except for positive conformal dilatations !

Let us recall that a *conformal dilatation* is a map that is conformally conjugate to an r-dilatation, i.e., a map $\varphi = \alpha^{-1} \circ \delta_r \circ \alpha$, where $r \in \mathbb{D}$ and α is a conformal automorphism of $\mathbb{D}$. Each such map fixes the point $p = \alpha^{-1}(0) \in \mathbb{D}$.

Theorem 5 *If ϕ is a holomorphic selfmap of $\mathbb{D}$ that fixes a non-zero point in $\mathbb{D}$ and is neither the identity map on $\mathbb{D}$ nor a positive conformal dilatation. Then* $0 \in \operatorname{int} W(C_\phi)$.

The special case of positive dilatations is treated in the following theorem.

Theorem 6 *If ϕ is a positive conformal dilatation that fixes a point $p \in \mathbb{D}$ and has dilatation parameter r ($0 < r < 1$). Then the following are equivalent:*

1. $0 \in W(C_\phi)$
2. $0 \in \operatorname{int} W(C_\phi)$
3. $|p| > \sqrt{r}$.

1.4 What about the point 1? [12]

Theorem 7 *Suppose ϕ is a holomorphic selfmap of $\mathbb{D}$.*

1. *$\phi(0) = 0$ if and only if the point 1 is an extreme boundary point of $W(C_\phi)$.*
2. *If $\phi(0) \neq 0$, then $1 \in \operatorname{int} W(C_\phi)$.*

2 Composition operators on $\mathscr{H}^2$

$\mathscr{H}^2$ is the Hilbert space of Dirichlet series with square-summable coefficients, equipped with the norm $\|f\| = \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}$ if $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ belongs to $\mathscr{H}^2$. By the Cauchy-Schwarz inequality, the functions in $\mathscr{H}^2$ are all holomorphic on the half-plane $\mathbb{C}_{1/2} = \{s \in \mathbb{C}, \Re s > 1/2\}$, with reproducing kernel $\mathscr{K}_a$ $(a \in \mathbb{C}_{1/2})$, i.e. $f(a) = \langle f, \mathscr{K}_a \rangle$ an $\mathscr{K}_a(s) = \zeta(s + \overline{a})$ where ζ denotes the Riemann Zeta-function.

For the space $\mathscr{H}^2$ not any analytic function in a half-plane can be represented as a Dirichlet series. Thus the situation is different than the one of H^2. The analog of the classical Littlewood subordination principle in the context of Dirichlet series is a theorem due to Gordon and Hedenmaln [11]. We denote by $\mathbb{C}_\theta$ the half-plane $\mathbb{C}_\theta = \{s \in \mathbb{C}, \Re s > \theta\}$.

Theorem 8 *An analytic self-map $\phi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ induces a bounded composition operator $C_\phi : f \mapsto f \circ \phi$ on $\mathscr{H}^2$ if and only if*

1. *ϕ is **"representable"** i.e., $\phi(s) = c_0 s + \varphi(s)$, where c_0 is a non-negative integer, and where the analytic function φ can be written as a convergent Dirichlet series $\sum_1^\infty c_n n^{-s}$ for $\Re s$ large enough: $\Re s > \theta$ (in short $\varphi \in \mathscr{D}$).*

2. *ϕ is **"extendable"** with **"controlled range"**, namely ϕ has an analytic extension to $\mathbb{C}_0$, still denoted by ϕ, and such that*

   (a) $\phi(\mathbb{C}_0) \subset \mathbb{C}_0$ *if* $c_0 \geq 1$.

   (b) $\phi(\mathbb{C}_0) \subset \mathbb{C}_{1/2}$ *if* $c_0 = 0$.

Let us mention that the cases $c_0 \geq 1$, $c_0 = 0$ are very different. Let us recall [9] the following:
If $\phi(s) = c_0 s + \varphi(s) : \mathbb{C}_0 \to \mathbb{C}_0$, $\varphi \in \mathscr{D}$, $\varphi(s) = \sum_1^\infty c_n n^{-s}$, then:

1. if $\varphi(s) = c_1$, we have $\Re c_1 \geq 0$,
2. if φ is not constant, we have $\Re c_1 > 0$.

In the case of H^2, C_ϕ is invertible if and only if ϕ is an automorphism of $\mathbb{D}$ (Theorem 1).
In the case of $\mathscr{H}^2$ (when C_ϕ is a bounded operator on $\mathscr{H}^2$), C_ϕ is invertible if and only if $\phi(s) = s + ik$, $k \in \mathbb{R}$.
Thus the situation is power in the case of $\mathscr{H}^2$ than in the case of H^2.
Let us recall the following results of F. Bayart [2].

Theorem 9 *For a bounded composition operator $C_\phi : \mathscr{H}^2 \to \mathscr{H}^2$, the following are equivalent:*

1. *C_ϕ is invertible,*
2. *C_ϕ is Fredholm,*
3. *$\phi(s) = s + ik$, where k is a real number.*

Theorem 10 *Let $C_\phi : \mathscr{H}^2 \to \mathscr{H}^2$ be a bounded composition operator. Then:*

1. *C_ϕ is normal if and only if $\phi(s) = s + c_1$, where $\Re c_1 \geq 0$.*
2. *Let $\phi(s) = c_0 s + \varphi(s)$. Assume that the Dirichlet series of φ converges uniformly for $\Re s \geq 0$, then C_ϕ is isometric if and only if $\phi(s) = c_0 s + ik$, where $c_0 \geq 1$ and $k \in \mathbb{R}$.*

2.1 Numerical range for symbols $\phi(s) = c_0 s + c_1$, $c_0 \in \mathbb{N}$, $\Re c_1 \geq 0$ [9]

1. $\phi(s) = s$
   $W(C_\phi) = \{1\}$

2. $\phi(s) = c_1 \in \mathbb{C}_{1/2}$
   $W(C_\phi)$ is the closed elliptic disk whose boundary is the ellipse of foci 0 and 1 having horizontal axis of length $\|\mathscr{K}_{c_1}\| = \left(\xi(2\Re c_1)\right)^{1/2}$.

3. $\phi(s) = s + c_1$, $c_1 \neq 0$

   (a) If $c_1 > 0$, $W(C_\phi) =]0,1]$. If $\Re c_1 > 0$ and $c_1 \notin \mathbb{R}$, $W(C_\phi)$ is a closed polygon containing the origin in its interior.

   (b) If $\Re c_1 = 0$, $W(C_\phi) = \mathbb{D} \cup \{n^{-c_1}, n \geq 1\}$.

4. $\phi(s) = c_0 s + c_1, c_0 \geq 2$

   (a) If $c_1 = ik$, $k \in \mathbb{R}$, one has $W(C_\phi) = \mathbb{D} \cup \{1\}$. And this remains true if ϕ is any symbol such that C_ϕ is a non-surjective isometry of $\mathscr{H}^2$ into itself.

   (b) If $\Re c_1 > 0$, one has $W(C_\phi) = \mathrm{co}\big(\overline{D}(0,r) \cup \{1\}\big)$, where $r < 1$ is given by the relation

$$r = \sup\left\{\sum_{h\geq 0} a_h a_{h+1} 2^{-c_0^h \gamma_1};\ a_h \geq 0,\ \sum_0^\infty a_h^2 = 1\right\}, \qquad \gamma_1 = \Re c_1. \tag{1}$$

   (c) In particular, we always have $0 \in \operatorname{int} W(C_\phi)$.

2.2 Zero containement [9]

As in the case of H^2 we get

Theorem 11 *For any symbol ϕ that is not the identity,* $0 \in \overline{W(C_\phi)}$

We consider as in the section 1.3 the following question: *For which ϕ does int $W(C_\phi)$ contain the origin ?*

We already know (section 2.1, 4) that if $\phi(s) = c_0 s + c_1$, $c_0 \geq 2$ then $0 \in \operatorname{int} W(C_\phi)$. In fact, we get

Theorem 12 *Let ϕ be the symbol of a composition operator on $\mathscr{H}^2$. Then*

1. *Either $\phi(s) = s + c_1$, $c_1 > 0$. Or 0 belongs to the interior of $W(C_\phi)$.*
2. *If $\phi(s) \neq s + c_1$, $c_1 > 0$, $W(C_\phi)$ is closed as soon as C_ϕ is compact.*

2.3 What about the point 1?

Let us recall that an eigenvalue λ of a bounded operator T on a Hilbert space is said to be *normal* if

$$\ker(T - \lambda I) = \ker(T^\star - \overline{\lambda} I).$$

Theorem 13 *Let $\phi(s) = c_0 s + \varphi(s)$, $\varphi \in \mathscr{D}$.*

1. *$c_0 \neq 0$ if and only if $1 \in \partial W(C_\phi)$.*
2. *If $c_0 = 0$ then $1 \in \operatorname{int} W(C_\phi)$.*

PROOF.

1. Let us recall that it is customary (see [14]) to say that ω, of modulus one, is a fixed point of $\phi : \mathbb{D} \to \mathbb{D}$ if $\lim_{r \overset{<}{\longrightarrow} 1} \phi(r\omega) = \omega$. Similarly, we will say here that ∞ is a fixed point of ϕ if $\lim_{\Re s \to +\infty} \phi(s) = \infty$. That is always the case when $c_0 \neq 0$, since $\varphi(s)$ is defined by an absolutely convergent (therefore bounded) Dirichlet series for $\Re s$ large enough. Then C_ϕ is a contraction [11] and $W(C_\phi) \subset \overline{\mathbb{D}}$, this implies $1 \in \partial W(C_\phi)$.

Let us now suppose that $1 \in \partial W(C_\phi)$.

As 1 is always an eigenvalue of C_ϕ and $1 \in \partial W(C_\phi)$, it follows [4] that 1 is a normal eigenvalue. It means that $C_\phi^\star 1 = 1$. One has $C_\phi^\star(\mathscr{K}_a) = \mathscr{K}_{\phi(a)}$; where $\mathscr{K}_a(s) = \zeta(\overline{a}+s) = \sum_{n\geq 1} n^{-\overline{a}-s}$. It is easy to see that

$$C_\phi^\star 1 = \lim_{\Re a \to +\infty} C_\phi^\star(\mathscr{K}_a) = \lim_{\Re a \to +\infty} \mathscr{K}_{\phi(a)} = 1.$$

This is only possible if $\lim\limits_{\Re a \to +\infty} \phi(a) = +\infty$.

2. It follows from [11] that, for $n \geq 2$, one has $n^{-\phi(s)} = n^{-c_1}\Big(1 + \sum\limits_{\ell \geq 2} d_\ell^{(n)} \ell^{-s}\Big)$.

Now consider $f(s) = a + b2^{-s}$, with $|a|^2 + |b|^2 = 1$. Then

$$C_\phi f(s) = a + b2^{-c_1}\Big(1 + \sum_{\ell \geq 2} d_\ell^{(2)} \ell^{-s}\Big).$$

And $\langle C_\phi f, f\rangle = |a|^2 + |b|^2 2^{-c_1} + b\overline{a} 2^{-c_1} d_2^{(2)}$.

Therefore, $W(C_\phi) \supset W(A)$, where A is the matrix $A = \begin{bmatrix} 1 & 2^{-c_1} d_2^{(2)} \\ 0 & 2^{-c_1} \end{bmatrix}$ on $\mathbb{C}^2$.

Then $W(A)$ is an non degenerated elliptic disk with foci 1 and 2^{-c_1} and $1 \in \operatorname{int} W(C_\phi)$. ■

Remark 1 *It can be shown that if C_ϕ is a non unitary isometry then $\overline{W(C_\phi)} = \overline{\mathbb{D}}$.*

3 Compactness — Fixed points

3.1 H^2-setting

Let us recall the following results

Theorem 14 ([14]) *Let ϕ be a holomorphic selfmap of $\mathbb{D}$. Then*

1. *If C_ϕ is compact, we have*

$$\lim_{|z|\to 1} \frac{1-|\phi(z)|}{1-|z|} = \infty.$$

2. *The converse of 1 is true if ϕ is injective, or finitely valent.*

3. *If ϕ has restricted range (i.e. $\|\phi\|_\infty < 1$), C_ϕ is compact, and even in any Schatten class S_p, $p > 0$. The converge is not true.*

Theorem 15 ([14]) *If C_ϕ is compact then ϕ has a fixed point in $\mathbb{D}$.*

3.2 $\mathscr{H}^2$-setting

By analogy with 3, and in view of the Gordon-Hedenmalm Theorem, we shall say that $\phi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$, giving rise to a bounded composition operator, has *restricted range* if

1. $c_0 \geq 1$ and $\phi(\mathbb{C}_0) \subset \mathbb{C}_\varepsilon$, for some $\varepsilon > 0$.

2. $c_0 = 0$ and $\phi(\mathbb{C}_0) \subset \mathbb{C}_{1/2+\varepsilon}$, for some $\varepsilon > 0$.

The following simple fact was observed by Bayart ([2]):
If ϕ has restricted range, then $C_\phi : \mathscr{H}^2 \to \mathscr{H}^2$ is compact.
But the converse is not true and has been studied in [10, 3] (see also [5, 13]).

We are yet far from being able to prove the existence of a fixed point for ϕ if $C_\phi : \mathscr{H}^2 \to \mathscr{H}^2$ is compact, and will content ourselves with the two following propositions.

Proposition 1 *Suppose $\phi(s) = c_0 s + \varphi(s)$, where:*

1. *$\phi(\mathbb{C}_0) \subset \mathbb{C}_\varepsilon$, for some $\varepsilon > 0$,*

2. *the Dirichlet series of φ converges uniformly in $\mathbb{C}_0$.*

Then ϕ has a fixed point.

PROOF. As mentioned before, when $c_0 \neq 0$, then ϕ has a fixed point. Therefore we only have to prove that ϕ has a fixed point when $c_0 = 0$.

It follows from the hypothesis (2) that $|\phi(s)| \leq M$ for $s \in \mathbb{C}_0$. Let now $h : \mathbb{C}_0 \to \mathbb{D}$ be the Cayley map defined by $h(z) = \frac{z-1}{z+1}$, and $\psi : \mathbb{D} \to \mathbb{D}$ be the conjugate of ϕ by h: $\psi = h \circ \phi \circ h^{-1}$. Observe that $1 - |h(z)|^2 = 4\Re z/|z+1|^2$, so that $1 - |h(\phi(s))|^2 \geq 4\varepsilon/(M+1)^2$ for $s \in \mathbb{C}_0$, and that ψ sends $\mathbb{D}$ into $\overline{D}_r = \{|z| \leq r\}$, where $r = \sqrt{1 - \frac{4\varepsilon}{(M+1)^2}} < 1$. Then, by well-known results (Brouwer's or Rouche's theorem), ψ has a fixed point $a \in \mathbb{D}$, and ϕ has the fixed point $h^{-1}(a) \in \mathbb{C}_0$. ■

If a is a fixed point of C_ϕ, then $C_\phi^\star(\mathscr{K}_a) = \mathscr{K}_{\phi(a)} = \mathscr{K}_a$, and $\mathscr{K}_a$ is a non-zero fixed vector of $C_\phi^\star$; by extension, we will say that ϕ has a weak fixed point if there exists a non-zero $f \in \mathscr{H}^2$ such that $C_\phi^\star(f) = f$.

If we make the stronger assumption that ϕ has restricted range (which guarantees the compactness of C_ϕ), we can get rid of the regularity assumption on φ, for a weaker conclusion.

Proposition 2 *Suppose that, for some $\varepsilon > 0$, we have $\phi(\mathbb{C}_0) \subset \mathbb{C}_{1/2+\varepsilon}$. Then, ϕ has a weak fixed point.*

PROOF. We will use the notations of Gordon and Hedenmalm ([11]); let $p_1 < \cdots < p_h < \cdots$ be the prime numbers; for $f(s) = \sum_1^\infty a_n n^{-s} = \sum_1^\infty a_n (p_1^{-s})^{\alpha_1} \cdots (p_r^{-s})^{\alpha_r}$ (where $n = p_1^{\alpha_1} \cdots p_2^{\alpha_r}$) $\in \mathscr{H}^2$, we write $\mathscr{D} f(z) = \sum_1^\infty a_n z_1^{\alpha_1} \cdots z_r^{\alpha_r}$ (formally), and it follows from a result (summation process) of Cole and Gamelion ([8]) that $\mathscr{D} f$ may be considered as an analytic function of infinitely many variables $z = (z_1, \ldots, z_r, \ldots)$ on the open set $\mathbb{D}^\infty \cap \ell_2$ of the Hilbert space ℓ_2, $\mathbb{D}^\infty$ denoting the infinite polydisk: $\mathbb{D}^\infty = \{z = (z_1, \ldots, z_r, \ldots);\ |z_j| < 1$ for each $j\}$. Also set $\phi_h(s) = p_h^{-\phi(s)}$, $\tilde{\phi}(z) = (\mathscr{D}\phi_1(z), \mathscr{D}\phi_2(z), \dots, \mathscr{D}\phi_h(z), \dots)$ for $z \in \mathbb{D}^\infty \cap \ell_2$ we have (see [8]): $\mathscr{D} C_\phi \mathscr{D}^{-1} = C_{\tilde{\phi}}$, where $\tilde{\phi}$ is an analytic self-map of $\mathbb{D}^\infty \cap \ell_2$.

Now fix an integer k, and consider the following diagram:

$$\mathbb{D}^h \xrightarrow{j_h} \mathbb{D}^\infty \cap \ell_2 \xrightarrow{\tilde{\phi}} \mathbb{D}^\infty \cap \ell_2 \xrightarrow{p_h} \mathbb{D}^h,$$

$\tilde{\phi}_h = p_h \circ \tilde{\phi} \circ j_h$, where j_h is the canonical injection and p_h the orthogonal projection; that is:

$$\tilde{\phi}_h(z_1, \dots, z_h) = \big(\mathscr{D}\phi_1(z_1, \dots, z_h, 0, \dots), \dots, \mathscr{D}\phi_h(z_1, \dots, z_h, 0, \dots), 0, \dots, 0, \dots\big).$$

Denote by Δ_h the compact subpolydisk of $\mathbb{D}^h$ defined by

$$\Delta_h = \big\{z = (z_1, \dots, z_h);\ |z_j| \leq p_j^{-1/2-\varepsilon} \text{ for } 1 \leq j \leq h\big\}.$$

From the assumption on ϕ, we see that $\tilde{\phi}_h$ maps Δ_h into itself, therefore has a fixed point $\big(a_1^{(h)}, \dots, a_h^{(h)}\big)$, by the Brouwer fixed point theorem. That is:

$$\begin{aligned} \big|a_j^{(h)}\big| &\leq p_j^{-1/2-\varepsilon}, \qquad \text{for } 1 \leq j \leq h \\ a_j^{(h)} &= \mathscr{D}\phi_j\big(a_1^{(h)}, \dots, a_h^{(h)}, 0, \dots\big) \qquad \text{for } 1 \leq j \leq h. \end{aligned} \tag{2}$$

Now, set $A = \big\{z = (z_1, \dots) \in \mathbb{D}^\infty \cap \ell_2;\ |z_j| \leq p_j^{-1/2-\varepsilon} \text{ for each } j\big\}$, and $a^{(h)} = \big(a_1^{(h)}, \dots, a_h^{(h)}, 0, \dots\big)$. Since $\sum_j \big(p_j^{-1/2-\varepsilon}\big)^2 < \infty$, A is a compact (of "Hilbert cube type") subset of $\mathbb{D}^\infty \cap \ell_2$, and $a^{(h)} \in A$, so (up to the extraction of a subsequence), we may assume that $a^{(h)}$ converges strongly to $a \in A$, and we get from (2) that $a_j = \mathscr{D}\phi_j(a)$ for each j, that is $\tilde{\phi}(a) = a$.

If K_a is the reproducing kernel of the functional Hilbert space $H^2\big(\mathbb{D}^\infty \cap \ell_2\big)$ at a, we then have $C_{\tilde{\phi}}^\star(K_a) = K_a$. None, $\mathscr{D}$ is unitary, so that $C_\phi = \mathscr{D} C_{\tilde{\phi}} \mathscr{D}^{-1}$ and $C_\phi^\star = \mathscr{D}^{-1} C_{\tilde{\phi}}^\star \mathscr{D}$. So that $C_\phi^\star(f) = f$, with $f = \mathscr{D}^{-1}(K_a) \neq 0$. This ends the proof of Proposition 2. ∎

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